TURBULENCE

CLASSIC PAPERS ON STATISTICAL THEORY

Edited by
S. K. FRIEDLANDER
THE JOHN HOPKINS UNIVERSITY
and
LEONARD TOPPER
U. S. ATOMIC ENERGY COMMISSION

INTERSCIENCE PUBLISHERS, INC., NEW YORK
INTERSCIENCE PUBLISHERS LTD., LONDON

Library of Congress Catalog Card Number 61-10829

Grateful acknowledgment is made to the following for permission to reprint the papers contained herein: The London Mathematical Society; The Royal Society (London); W. Prager, editor of the *Quarterly of Applied Mathematics; Journal of Marine Research* of the Foundation for Marine Research; the National Academy of Sciences; E. U. Condon, editor of the *Reviews of Modern Physics;* and to the authors of the individual papers.

Interscience Publishers, Inc., 250 Fifth Avenue, New York 1, N. Y.

For Great Britain and Northern Ireland:

Interscience Publishers Ltd., 88/90 Chancery Lane, London, W. C. 2

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

The group of papers in this collection represent significant steps in the development of the statistical theory of turbulence. They were all published prior to 1950; at that time our physical understanding of turbulence was already adequate for the study of many related scientific questions. The concepts of the correlation coefficient, spectrum function, and local similarity which had proved so helpful in revealing the structure of turbulence had been developed. Further advances have been at the expense of considerably greater effort.

The statistical theory may have begun with the Taylor paper of 1921, in which the concept of the Lagrangian correlation coefficient was advanced and which provided a theoretical basis for turbulent diffusion. It was not until 1935, however, that the approach, in a revised form, was taken up again by Taylor who defined many of the properties of isotropic turbulence. This was followed by the von Kármán and Howarth (1938) demonstration of the tensor properties of the Eulerian correlation coefficient and their formulation of the relationship between the double and triple coefficients through the Navier-Stokes equations. At about the same time, Taylor (1938) showed the transform relationship between the correlation coefficient and the energy spectrum function. A review of the theory up to the early 40's with some interesting applications to problems of diffusion is given by Dryden (1943). The Taylor concept of the energy spectrum has dominated turbulence research from this period to the present. The similarity hypothesis of Kolmogoroff (1941) served as the starting point of many investigations and led to general theories of spectral similarity such as those proposed by von Kármán and Lin (1948, 1949).

The collection should serve as a convenient reference to what is now classical literature for those interested in fluid turbulence in itself. Workers in other fields, some peripheral and others rather far afield from turbulence, find the methods developed in the statistical theory increasingly useful. For them, and the editors include themselves among this group, it is hoped that this volume will serve as a primary reference. Some of the peripheral problems which have already been attacked using these approaches include the dispersion of particles, drop break-up, particle coagulation, mixing with and without chemical reaction, the spectra of surface waves and solar electrodynamics. As yet little studied are heat and mass transfer to particles suspended in a turbulent fluid.

The theoretical methods developed for analyzing the turbulence spectrum will also have application to other spectral problems such as particle size distributions in atmospheric aerosols and clouds. Here, again, we encounter the phenomenon of transfer (of matter in this case) to the upper end of the size spectrum by a non-linear interaction among the components at the lower end. Again, there may be "dissipation" in the form of local sedimentation or drop break-up.

A brief bibliography of recent theoretical papers dealing with these applications of the statistical theory follows:

Particle Mechanics

1. Tchen, *Mean Value and Correlation Problems Connected with the Motion of Small Particles Suspended in a Turbulent Fluid,* Martinus Nijhoff, The Hague, 1947.
2. Corrsin, S., and Lumley, J., "On the Equation of Motion for a Particle in a Turbulent Fluid," *App. Sci. Research* **A6,** 114 (1956).
3. Friedlander, S. K., "Behavior of Suspended Particles in a Turbulent Fluid," *A. I. Ch. E. Journal* 3, 381 (1957).

Drop Break-up

1. Kolmogoroff, A. N., "On the Disintegration of Drops in Turbulent Flow" (*in Russian*), *Doklady Akad. Nauk* S.S.S.R. **66,** 825 (1949).
2. Hinze, J. O., "Fundamentals of the Hydrodynamic Mechanism of Splitting in Dispersion Processes," *A. I. Ch. E. Journal* 1, 289 (1955).

Particle Coagulation

1. Levich, B. L., "Theory of Colloid Coagulation in a Turbulent Fluid Stream" (*in Russian*), *Doklady Akad. Nauk* S.S.S.R. **99,** 809 (1954).
2. Saffman, P. G., and Turner, J. S., "On the Collision of Drops in Turbulent Clouds," *J. Fluid Mech.* 1, 16 (1956).

Mixing and Reaction

1. Corrsin, S., "On the Spectrum of Isotropic Temperature Fluctuations in an Isotropic Turbulence," *J. Appl. Phys.* **22,** 469 (1951).

2. CORRSIN, S., "Statistical Behavior of a Reacting Mixture in Isotropic Turbulence," *Physics of Fluids* **1**, 42 (1958).
3. BATCHELOR, G. K., "Small-Scale Variation of Convected Quantities Like Temperature in Turbulent Fluid," Parts 1 and 2, *J. Fluid. Mech.* **5**, 113 (1959).

Surface Wave Spectra

1. PHILLIPS, O. M., "The Equilibrium Range in the Spectrum of Wing-Generated Waves," *J. Fluid Mech.* **4**, 426 (1958).

Heat and Mass Transfer to Particles

1. TUNITZKII, N. N., "On Diffusional Process in Conditions of Natural Turbulence" (*in Russian*), *J. Phys. Chem. (U.S.S.R.)* **20**, 1137 (1946).

Particle Size Spectra

1. FRIEDLANDER, S. K., "On the Particle Size Spectrum of Atmospheric Aerosols," *J. Meteor.* **17**, 373 (1960).

Solar Physics

1. COWLING, T. G., *Magnetohydrodynamics*, Interscience Publishers, Inc., New York, 1957, especially pp. 92–98.

CONTENTS

DIFFUSION BY CONTINUOUS MOVEMENTS

By G. I. TAYLOR.

[Received May 22nd, 1920.—Read June 10th, 1920.]

Introduction.

It has been shown by the author,* and others, that turbulent motion is capable of diffusing heat and other diffusible properties through the interior of a fluid in much the same way that molecular agitation gives rise to molecular diffusion. In the case of molecular diffusion the relationship between the rate of diffusion and the molecular constants is known; a large part of the Kinetic Theory of Gases is devoted to this question. On the other hand, nothing appears to be known regarding the relationship between the constants which might be used to determine any particular type of turbulent motion and its "diffusing power."

The propositions set down in the following pages are the result of efforts to solve this problem.

In order to simplify matters as much as possible the transmission of heat in one direction only, that of the axis of x, will be considered. We shall take the case of an incompressible fluid whose temperature θ, at time $t = 0$, depends only on x, and increases or decreases uniformly with x. Initially therefore $\partial\theta/\partial x$ is constant and equal to β, say.

Now suppose that the fluid is moving in turbulent motion, so that the distribution of temperature is continually altering. Suppose that the turbulent motion could be defined by means of the Lagrangian equations of fluid motion, so that the coordinates (x, y, z) of a particle are given in terms of its initial coordinates (a, b, c) at the time $t = 0$, and of t.

Since the temperature of any particle is supposed to remain constant during the motion, the temperature at the point (x, y, z) at time t, which will be represented by the symbol $\theta(x, y, z)$ is $\theta(a, 0)$, which represents the temperature at $x = a$ at time $t = 0$.

---

* "Eddy Motion in the Atmosphere," *Phil. Trans.*, 1915, p. 1.

Since the rate of increase in temperature with x constant when $t=0$,

$$\theta(a, 0) = \theta(x, 0) - (x-a)\beta.$$

The average rate at which heat is being conveyed across unit area of a plane perpendicular to the axis of x is evidently equal to $-\rho\sigma\beta$ multiplied by the average value of $u(x-a)$ over a large area of a plane perpendicular to the axis of x. In these expressions u represents the velocity of a particle of fluid in the direction of the axis of x, ρ is the density, and σ the specific heat, so that $\rho\sigma$ is the heat capacity of unit volume of the fluid.

No doubt the average value of $u(x-a)$, which must be obtained from considerations of the particular nature of the turbulent motion in question, depends on the mean motion of the fluid; but if experimental data exist, as in fact they do, which enable its value to be calculated, it is of interest to enquire what types of turbulent motion are capable of producing the observed distribution of temperature.

In order to simplify matters still further it will be assumed that the turbulent motion is uniformly distributed throughout space. The mean value of $u(x-a)$ will then be the same for every layer and will be equal to the mean value throughout space. This quantity will be expressed by the symbol $[u(x-a)]$.

Owing to the fact that the fluid is incompressible $[u(x-a)]$ could be calculated either by taking a rectangular element $\delta x\,\delta y\,\delta z$, at time t, finding the corresponding value of $u(x-a)$ and integrating throughout space; or by taking an element $\delta a\,\delta b\,\delta c$ at time $t=0$, finding the corresponding value of $u(x-a)$ at time t, and integrating. The second method will be adopted.

Fixing our attention on a particle of fluid, it will be noticed that

$$u = \frac{\partial x}{\partial t} \quad \text{and} \quad x-a = \int_0^t u\,dt.$$

Hence, writing X for $x-a$,

$$[u(x-a)] = \left[X\frac{dX}{dt}\right] = \tfrac{1}{2}\left[\frac{dX^2}{dt}\right] = \tfrac{1}{2}\frac{d}{dt}[X^2].$$

In this ideally simplified system therefore the rate at which heat is transferred in the direction of the axis of x is determined by the rate of increase of the mean value of the square of the distance, parallel to the axis of x, which is moved through by a particle of fluid in time t.

If a physicist were to try to define the characteristic features of any particular case of turbulent motion, with a view to discussing statistically its effect as a conductor of heat, he would probably first fix his attention on the mean energy of the motion. That is to say, he would determine $[u^2]$.

He would then perhaps notice that it is not sufficient to determine $[u^2]$. With a given value of $[u^2]$ it is possible for the turbulent motion to be associated with a small or a large transfer of heat, according to whether a particle frequently, or infrequently, reverses its direction of motion. It would therefore be necessary to define some characteristic of the motion which differentiates between the cases in which the changes in the velocity of a particle are rapid, and those in which they are slow. A suitable characteristic to choose would be $\left[\left(\frac{du}{dt}\right)^2\right]$.

Further investigation would show that it is necessary also to define

$$\left[\left(\frac{d^2u}{dt^2}\right)^2\right], \ \ldots, \ \left[\left(\frac{d^nu}{dt^n}\right)^2\right], \ \ldots .$$

The relationship between

$$\tfrac{1}{2}\frac{d}{dt}[X^2] \text{ and } [u^2], \quad \left[\left(\frac{du}{dt}\right)^2\right], \ \ldots, \ \left[\left(\frac{d^nu}{dt^n}\right)^2\right], \ \ldots$$

is discussed in the following pages. The problem is in some respects similar to that known as "The drunkard's walk," or to Karl Pearson's* problem of the random migration of insects, when the motion is limited to one dimension; but in the course of the investigation some curious propositions have come to light concerning the mean values of continuously varying quantities which may perhaps be of interest to mathematicians, as well as to physicists.

In the course of the work no discussion of the convergency of the series used is attempted. The work must therefore be regarded as incomplete. The author feels that such questions might be examined with advantage by a pure mathematician, and it is in the hope of interesting one of them that he wishes to offer this paper to the London Mathematical Society.

Discontinuous Motion.

Before proceeding to consider the continuous version of the problem of random migration in one dimension, the discontinuous case will be

---

* Drapers' Company *Memoirs*.

extended slightly, so as to make it bear some resemblance to the continuous case.

Suppose that a point starts moving with uniform velocity v along a line, and that after a time τ it suddenly makes a fresh start and either continues moving forward with velocity v or reverses its direction and moves back over the same path with the same velocity v. Suppose that this process is repeated n times and that we consider the mean values of the quantity concerned for a very large number of such paths.

Let x_r be the distance moved over in the r-th interval. Then x_r is numerically equal to $v\tau$, but its sign may be either positive or negative and each occurs an equal number of times in considering the average. If X_n is the standard deviation or "root mean square" of the distance moved by the point from the original position after time $n\tau$, then

$$X_n^2 = [(x_1+x_2+x_3+\ldots+x_n)^2],$$

where the square bracket indicates that the mean value is taken for all the paths.

Hence $$X_n^2 = nd^2+2[x_1x_2+x_1x_3+\ldots+x_rx_s+\ldots], \tag{1}$$

where $$d = v\tau.$$

If there is no correlation between any two x's,

$$[x_rx_s] = 0.$$

Hence $$X_n^2 = nd^2, \quad \text{or} \quad X_n = d\sqrt{n} = v\sqrt{(\tau T_n)},$$

where T_n is the total time during which the migration has been taking place. It will be seen therefore that X_n is proportional to $\sqrt{T_n}$.

Actually in a turbulent fluid or in any continuous motion there is necessarily a correlation between the movement in any one short interval of time and the next. This correlation will evidently increase as the interval of time diminishes, till, when the time is short compared with the time during which a finite change in velocity takes place, the coefficient of correlation tends to the limiting value unity.

This idea will now be introduced into equation (1).

To begin with let us make the arbitrary assumption that x_r is correlated with x_{r+1} by a correlation coefficient c. Suppose also that the partial correlations of x_r with x_{r+2}, x_{r+3}, ... are all zero. The correlation coefficient between x_r and x_{r+2} is then c^2. Between x_r and x_{r+s} it is c^s.

The value of $2[x_1x_2+x_1x_3+\ldots+x_rx_s+\ldots]$ is then

$$2d^2\{nc+(n-1)c^2+(n-2)c^3+\ldots+c^n\}.$$

The series in the { } bracket is easily summed. Substituting its value in (1) it will be found that

$$X_n^2 = d^2 \left\{ n + \frac{2nc}{1-c} - \frac{2c^2(1-c^n)}{(1-c)^2} \right\},$$

or, putting $n = T_n/\tau$, and $d = v\tau$,

$$X_n^2 = v^2 \left\{ \left(\frac{1+c}{1-c}\tau\right) T_n - \frac{2c^2(1-c^n)\tau^2}{(1-c)^2} \right\}. \tag{2}$$

By reducing τ indefinitely we can evidently make the case approximate to some sort of continuous migration, but in order that X_n, v and T_n may be finite and tend to a definite limit as τ is decreased, it is necessary that $\left(\frac{1+c}{1-c}\right)\tau$ and $\frac{2c^2(1-c^n)\tau^2}{(1-c)^2}$ must also tend to a definite limit. That is to say, $1-c$ must be proportional to τ.

Let $\frac{\tau}{1-c}$ tend to the limit A when τ and $1-c$ tend to zero.

Then X_n^2 tends to the limiting value

$$v^2 \{2AT_n - 2A^2(1-e^{-T_n/A})\},$$

or, dropping the suffixes which are no longer necessary,

$$\sqrt{[X^2]} = v\sqrt{\{2AT - 2A^2(1-e^{-T/A})\}}, \tag{3}$$

where X is the distance traversed by a particle during a flight extending over an interval of time T, and the "root mean square" is taken for a large number of such flights.

When T is small this reduces to $\sqrt{[X^2]} = vT$, which is exactly what we should expect when the time is so short that the correlation coefficient c^n, between the first and last small element of migration has not fallen appreciably away from unity.

When T is large $\sqrt{[X^2]} = v\sqrt{(2AT)}$, so that the amount of "diffusion" is proportional to the square root of the time. The constant A evidently measures the rate at which the correlation coefficient between the direction of an infinitesimal path in the migration and that of an infinitesimal path at a time T, say, later, falls off with increasing values of T.

We have now seen how it is possible by introducing the idea of a correlation between the directions of the successive jumps in a random migration, to keep the standard deviation of the distance of migration constant, no matter how small the infinitesimal paths of the migration may be.

The migration is still a discontinuous one however. It suffers also from the disadvantage of depending on a special assumption, namely, that there is a definite correlation between the direction of motion in one infinitesimal element of path, and that in its immediate neighbours, but that there is no partial correlation between the directions of motion in paths which are not neighbours. This means that there is a special law of correlation between the directions of the paths at finite intervals of time. The correlation coefficient between the direction of an infinitesimal path and that of the path which occurs at a time $T = n\tau$ later, is evidently c^n. This may be written

$$\{1-(1-c)\}^n = (1-\tau/A)^n = (1-\tau/A)^{T/\tau}.$$

When τ is small this tends to the limit $e^{-T/A}$. (4)

We are therefore limiting ourselves to the particular type of motion in which the direction of an infinitesimal path is correlated to that at time T later by the correlation coefficient* $e^{-T/A}$.

Diffusion by continuous Movements.

The work just described, though not particularly useful for our present purpose, is useful in that it gives rise to ideas about how problems of migration or diffusion by continuous movements may be treated. In what follows these ideas are worked out and the conditions of motion which determine the laws of diffusion are found.

Before proceeding to discuss diffusion, however, it will be necessary to prove a few statistical properties of continuously varying quantities.

Suppose that we wish to express the characteristic properties of the variations of some quantity which varies continuously, but which appears to have no very definite law of variation. Suppose, for instance, it is desired to define the characteristic features of a barograph record. There are no obvious periods, nor is there any definite constant amplitude of variation in barometric pressure, yet there are certain properties of the curve which can be defined. If we take the standard deviation of pressure from its mean value during a year, it will be found to be practically constant from year to year. If p represents the deviation from the mean pressure, this standard deviation is $\sqrt{[p^2]}$, where the square bracket now indicates that the mean value of p^2 has been taken over a long period of

---

* Incidentally it will be noticed that the correlation between the direction of motion at one instant and that at time t *earlier* is also $e^{-T/A}$. It is obvious that we cannot consider the value of the expression $e^{-T/A}$ when T is negative.

time. One property of the curve which we can define, therefore, is the constancy of $\sqrt{[p^2]}$ during successive long periods.

The statistical properties of the barograph curve are by no means completely determined by this. It is possible to imagine an infinite variety of barograph curves with a given standard deviation of p. They might, for instance, have a large number of peaks in the curve during a given interval of time or a small number. In the former case the standard deviation of dp/dt might be expected to be larger than in the latter. We can, therefore, define the curve still further by specifying the standard deviations of dp/dt.

It appears that, from a given barograph curve, it is theoretically possible to find the standard deviations of p, dp/dt, $d^2p/dt^2, \ldots, d^np/dt^n, \ldots$. Let us assume that all these are constant from year to year.

Now suppose that we begin by specifying certain arbitrary standard deviations for p, dp/dt, &c., and that we try to construct a possible barograph curve from them. We are at once brought up against a difficulty. Suppose that we have specified a large number for the standard deviation of dp/dt, *i.e.* $\sqrt{[(dp/dt)^2]}$ and small numbers for $\sqrt{[p^2]}$ and $\sqrt{[(d^2p/dt^2)^2]}$. It is evident that if we begin constructing the curve with a large value of dp/dt at a point where $p = 0$, the fact that the value of $\sqrt{[(d^2p/dt^2)^2]}$ is small means that it will be a long time before dp/dt changes sign. Hence it will be a long time before p attains its maximum value, and during that time p must have attained a large value. Hence, if the standard deviation of dp/dt is large and that of d^2p/dt^2 is small, the standard deviation of p must be large. It is evident therefore that there must be some relationships between the standard deviations and the curve of which we have not yet taken account. We shall now see what these are.

Suppose that we observe the values $p_1, p_2, p_3, \ldots, p_n$ of p at a large number of successive times $t_1, t_2, t_3, \ldots, t_n$. Suppose further that we observe the values $p_1+\delta p_1$, $p_2+\delta p_2$, $p_3+\delta p_3$, $\ldots$, $p_n+\delta p_n$, at times $t_1+\delta t$, $t_2+\delta t$, $t_3+\delta t$, $\ldots$, $t_n+\delta t$, where δt is a small interval of time. Then, if $t_1, t_2, \ldots, t_n$ are taken at random

$$[p^2] = (p_1^2+p_2^2+\ldots+p_n^2)/n,$$

and since we are considering a curve in which $[p^2]$ is constant, $[p^2]$ is also, to the first order, equal to

$$\frac{1}{n}\left\{\left(p_1+\frac{dp_1}{dt}\,\delta t\right)^2+\left(p_2+\frac{dp_2}{dt}\,\delta t\right)^2+\ldots+\left(p_n+\frac{dp_n}{dt}\,\delta t\right)^2\right\}$$

$$= [p^2]+2\left[p\,\frac{dp}{dt}\right]\delta t.$$

It appears, therefore, that we can differentiate the quantities inside square brackets which indicate a mean value.

Hence the condition that $[p^2]$ shall be a constant is

$$\left[p\frac{dp}{dt}\right]=0. \tag{5}$$

There is, therefore, no correlation between p and dp/dt.

Now differentiate (5) once more,

$$\left[p\frac{d^2p}{dt^2}\right]+\left[\left(\frac{dp}{dt}\right)^2\right]=0. \tag{6}$$

Hence by the definition of a correlation coefficient, there is a negative correlation between p and d^2p/dt^2 equal to

$$\nu=-\frac{\left[\left(\frac{dp}{dt}\right)^2\right]}{\sqrt{[p^2]}\sqrt{\left[\left(\frac{d^2p}{dt^2}\right)^2\right]}}. \tag{7}$$

A consequence of the existence of this correlation coefficient ν is evidently that $[(dp/dt)^2]$ cannot be greater than $\sqrt{[p^2]}\sqrt{[(d^2p/dt^2)^2]}$, a statement which agrees with the remarks above.

The way in which the correlation coefficient affects the characteristic features of the p, t curve is easily seen. Suppose it is large, *i.e.* nearly equal to -1; then the curve will look something like curve (*a*), Fig. 1.

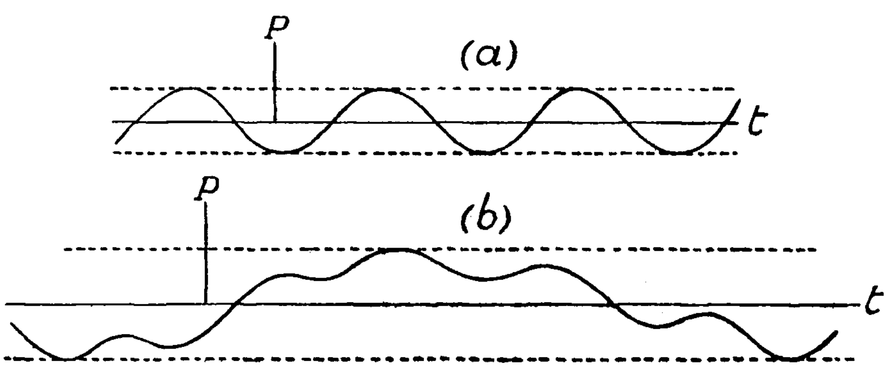


Fig. 1.

Suppose the correlation coefficient between p and d^2p/dt^2 is small, but that the standard deviations of dp/dt and d^2p/dt^2 are the same as in curve (*a*), Fig. 1, then the slopes and curvatures will be of the same magnitude as in curve (*a*), but the curvature will not always be concave to the mean line. This is shown in curve (*b*), Fig. 1.

It is evident that the standard deviation of p is greater in (*b*) than it is in (*a*). This is expressed by the formula (7), for if the standard deviations of dp/dt and d^2p/dt^2 are fixed, then the standard deviation of p is, according to (7), inversely proportional to ν.

Since the standard deviation of dp/dt has also been given as constant it can be treated exactly in the same way as the standard deviation of p, thus differentiating $[(dp/dt)^2]$, we have

$$\left[\frac{dp}{dt}\,\frac{d^2p}{dt^2}\right] = 0, \tag{8}$$

and differentiating this again

$$\left[\frac{dp}{dt}\,\frac{d^3p}{dt^3}\right] + \left[\left(\frac{d^2p}{dt^2}\right)^2\right] = 0. \tag{9}$$

But differentiating (6) again

$$\left[p\,\frac{d^3p}{dt^3}\right] + 3\left[\frac{dp}{dt}\,\frac{d^2p}{dt^2}\right] = 0.$$

Hence, from (8),

$$\left[p\,\frac{d^3p}{dt^3}\right] = 0. \tag{10}$$

Differentiating (10),

$$\left[p\,\frac{d^4p}{dt^4}\right] + \left[\frac{dp}{dt}\,\frac{d^3p}{dt^3}\right] = 0.$$

Hence, from (9),

$$\left[p\,\frac{d^4p}{dt^4}\right] - \left[\left(\frac{d^2p}{dt^2}\right)^2\right] = 0.$$

Proceeding in this way it can be shown that

$$\left.\begin{aligned} \left[p\,\frac{d^{2n}p}{dt^{2n}}\right] &= (-1)^n\left[\left(\frac{d^np}{dt^n}\right)^2\right], \\ \text{and} \qquad \left[p\,\frac{d^{2n+1}p}{dt^{2n+1}}\right] &= 0. \end{aligned}\right\} \tag{11}$$

The correlations to which p and its differential coefficients must be subject in order that their standard deviations may be constant, have now been established. We can, therefore, now use these standard deviations to define some statistical properties of the curve.

In analysing any actual curve, it may be very difficult and tedious to obtain these standard deviations. There is, however, another method of defining the statistical properties of the curve which is equivalent to that given above, but which is likely to be much more manageable in practice. This method will now be considered.

Suppose that we take, as before, the values $p_1, p_2, p_3, \ldots, p_n$, of p at a large number of times $t_1, t_2, t_3, \ldots, t_n$, chosen at random. Let us correlate them with the values $p_1', p_2', \ldots, p_n'$, of p at times $t_1+\xi, t_2+\xi, \ldots, t_n+\xi$, where ξ is a finite interval of time which may be positive or negative. Let the coefficient of correlation so found be R_ξ. Then R_ξ must evidently be a function of ξ.

If p_t be the value of p at time t, and $p_{t+\xi}$ be the value of p at time $t+\xi$, then by definition

$$[p_t p_{t+\xi}] = R_\xi \sqrt{[p_t^2]}\sqrt{[p_{t+\xi}^2]};$$

but by hypothesis the standard deviation of p does not vary, hence

$$[p_t^2] = [p^2] = [p_{t+\xi}^2],$$

and

$$R_\xi = [p_t p_{t+\xi}]/[p^2]. \tag{12}$$

Now expand $p_{t+\xi}$ in powers of ξ,

$$p_{t+\xi} = p_t + \xi\frac{dp}{dt} + \frac{\xi^2}{2!}\frac{d^2p}{dt^2} + \ldots.$$

Hence

$$[p_t p_{t+\xi}] = [p_t^2] + \xi\left[p\frac{dp}{dt}\right] + \frac{\xi^2}{2!}\left[p\frac{d^2p}{dt^2}\right] + \frac{\xi^3}{3!}\left[p\frac{d^3p}{dt^3}\right] \tag{13}$$

Substituting for $\left[p\dfrac{d^n p}{dt^n}\right]$ from (11), (13) becomes

$$[p_t p_{t+\xi}] = [p^2] + \xi(0) - \frac{\xi^2}{2!}\left[\left(\frac{dp}{dt}\right)^2\right] + \frac{\xi^3}{3!}(0) + \frac{\xi^4}{4!}\left[\left(\frac{d^2p}{dt^2}\right)^2\right]\ldots.$$

Hence, from (12),

$$R_\xi = 1 - \frac{\xi^2}{2!}\frac{\left[\left(\frac{dp}{dt}\right)^2\right]}{[p^2]} + \frac{\xi^4}{4!}\frac{\left[\left(\frac{d^2p}{dt^2}\right)^2\right]}{[p^2]} - \ldots + (-1)^n\frac{\xi^{2n}}{2n!}\frac{\left[\left(\frac{d^np}{dt^n}\right)^2\right]}{[p^2]}. \tag{14}$$

It will be seen that, as might have been expected, R_ξ is an even function of ξ.

As an example of the method let us take the case where it is known that $p = \sin(t+\epsilon)$, where ϵ may take all possible values, all of which are equally probable. In this case

$$[p^2] = \tfrac{1}{2}, \quad [(dp/dt)^2] = \tfrac{1}{2}, \quad \ldots, \quad [(d^np/dt^n)^2] = \tfrac{1}{2}, \quad \ldots,$$

(14) therefore becomes

$$R_\xi = 1 - \frac{\xi^2}{2!}\frac{(\frac{1}{2})}{(\frac{1}{2})} + \frac{\xi^4}{4!}\frac{(\frac{1}{2})}{(\frac{1}{2})} - \dots .$$

This is the series for $\cos\xi$. Hence $R_\xi = \cos\xi$. The correlation between the value of p at any time and its value when t is increased by any odd multiple of $\frac{1}{2}\pi$ is 0. This is obviously true since there is no correlation between $\sin(t+\epsilon)$ and $\sin\{t+\epsilon+(2n+1)(\frac{1}{2}\pi)\}$ as ϵ varies.

The correlations between p and its differential coefficients given in (11) are evidently also true.

Application to Diffusion by continuous Movements.

The theorems which have just been proved will now be used to find out what are the essential properties of the motion of a turbulent fluid which makes it capable of diffusing through the fluid properties such as temperature, smoke content, colouring matter or other properties which adhere to each particle of the fluid during its motion.

Consider a condition in which the turbulence in a fluid is uniformly distributed so that the average conditions of every point in the fluid are the same. Let u be the velocity parallel to a fixed direction, which we will call the axis of x, of the particle on which our attention is fixed. It will now be shown that the statistical properties which were defined above (now in relation to u instead of p) are sufficient to determine the law of diffusion, *i.e.* the law which governs the average distribution of particles initially concentrated at one point, at any subsequent time.

Suppose that the statistical properties of u are known in the form given above, that is to say, suppose that $[u^2]$ and R_ξ are known. R_ξ is now the correlation coefficient between the value of u for a particle at any instant, and the value of u for the same particle after an interval of time ξ.

Let u_t represent the value of u at time t. Consider the value of the definite integral

$$\int_0^t [u_t u_\xi]\, d\xi.$$

By the definition of R_ξ this is equal to

$$[u_t^2]\int_0^t R_{\xi-t}\, d\xi.$$

Hence, since $[u^2]$ does not vary with t, and R_ξ is an even function of ξ,

$$\int_0^t [u_t u_\xi]\, d\xi = [u^2] \int_0^t R_\xi\, d\xi. \tag{15}$$

Evidently one can integrate inside the square bracket just as one can differentiate. Hence

$$\int_0^t [u_t u_\xi]\, d\xi = \left[u_t \int_0^t u_\xi\, d\xi\right] = [u_t X],$$

or, in the notation of the introduction, $[uX]$.

Hence
$$[u^2] \int_0^t R_\xi\, d\xi = [uX] \tag{16}$$

$$= \tfrac{1}{2} \frac{d}{dt} [X^2], \tag{17}$$

and
$$[X^2] = 2\,[u^2] \int_0^T \int_0^t R_\xi\, d\xi\, dt, \tag{18}$$

where X is the distance traversed by a particle in time T.

Equation (18) is rather remarkable because it reduces the problem of diffusion, in a simplified type of turbulent motion, to the consideration of a single quantity, namely, the correlation coefficient between the velocity of a particle at one instant and that at a time ξ later.

Let us now consider the physical meaning of (18), when T is so small that R_ξ does not differ appreciably from 1 during the interval T. In this case

$$\int_0^T \int_0^t R_\xi\, d\xi\, dt = \tfrac{1}{2} T^2,$$

so that (18) becomes
$$[X^2] = [u^2]\, T^2,$$

or
$$\sqrt{[X^2]} = T \sqrt{[u^2]}. \tag{19}$$

That is to say, the standard deviation of a particle from its initial position is proportional to T when T is small. This is what we should expect provided the time T is so small that the velocity does not alter appreciably while the particle is moving over the path.

Now consider how one would anticipate that R_ξ would vary with ξ in a turbulent fluid. The most natural assumption seems to be that R_ξ would fall to zero for large values of ξ. It might remain positive as in the curve shown in Fig. 2, or it might become negative or oscillate before

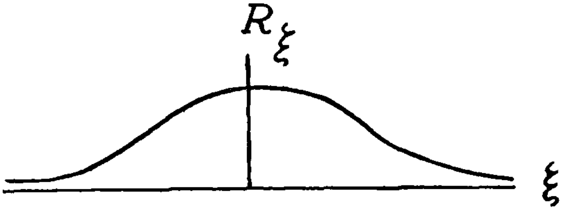


FIG. 2.

falling off to zero. In either case it seems probable that it will be possible to define an interval of time T_1, such that the velocity of the particle at the end of the interval T_1 has no correlation with the velocity at the beginning. In this case suppose that $\lim_{t\to\infty}\int_0^t R_\xi d\xi$ is finite and equal to I. Then at time $T\,(> T_1)$ after the beginning of the motion

$$\frac{d}{dt}[X^2] = 2\,[u^2]\,I,$$

so that $[X^2]$ increases at a uniform rate. In the limit when $[X^2]$ is large

$$\sqrt{[X^2]} = \sqrt{(2IT\,[u^2])}, \qquad (20)$$

so that the standard deviation of X is proportional to the square root of the time.

This, therefore, is a property which a continuous eddying motion may be expected to have which is exactly analogous to the properties of discontinuous random migration in one dimension.

It will be noticed that when $T > T_1$,

$$[Xu] = [u^2]\,I. \qquad (21)$$

Hence $[Xu]$ is constant in spite of the fact that $[X^2]$ continually increases. In order that this may be the case X must always be positively correlated with u, but the correlation coefficient must decrease with increasing $[X^2]$. If ν_{Xu} represents the correlation coefficient between X and u

$$\nu_{Xu} = -\frac{[Xu]}{\sqrt{[X^2]}\sqrt{[u^2]}} = \frac{I\sqrt{[u^2]}}{\sqrt{[X^2]}},$$

and in the limit when $T\to\infty$,

$$\nu_{Xu} = \frac{I\sqrt{[u^2]}}{\sqrt{(2IT[u^2])}} = \sqrt{\frac{I}{2T}}. \qquad (22)$$

It is interesting to compare the expression (18) for $[X^2]$ with the expression given in (3) for the standard deviation of X in the special case of discontinuous motion considered there.

In that case R_ξ was shown in (4) to be $e^{-\xi/A}$. In the continuous case if we write $R_\xi = e^{-\xi/A}$, (21) becomes

$$[X^2] = 2[u^2]\int_0^T\int_0^t e^{-\xi/A}\,d\xi\,dt$$

$$= 2[u^2]\int_0^T A(1-e^{-t/A})\,dt$$

$$= 2[u^2]\,\{AT-A^2(1-e^{-T/A})\}. \tag{23}$$

In the discontinuous case it was shown in (3) that

$$\sqrt{[X^2]} = v\sqrt{\{2AT-2A^2(1-e^{-T/A})\}}.$$

It is evident that this is exactly the same as (23) except that $\sqrt{[u^2]}$ has been substituted for the constant v which occurred in the discontinuous case.

If as a result of experiments on diffusion, it were possible to obtain a curve representing $[X^2]$ as a function of T, it would be possible to use (18) as a means of discovering something about the nature of the turbulence, for (18) could be written

$$\frac{d^2}{dt^2}[X^2] = 2[u^2]R_\xi,$$

and R_ξ could therefore be found.

In a recent communication to the Royal Society,* Mr. L. F. Richardson has described some experiments on the diffusion of smoke emitted from a fixed point in a wind. Similar observations have been made on the smoke from factory chimneys by Mr. Gordon Dobson.† Both these observers came to the conclusion that, at small distances from the origin of the smoke, the surface containing the standard deviations of the smoke from a horizontal straight line to leeward of the source, is a cone. If the mean velocity of the wind is assumed to be uniform, the standard deviation in a short interval of time is therefore proportional to the time. At greater distances their observations indicate that this surface becomes like a paraboloid, so that the deviation of the smoke is proportional to the square root of the time.

* *Phil. Trans.*, A, Vol. 221, p. 1.

† Advisory Committee for Aeronautics (*Reports*, 1919).

Both these observational data are in agreement with equations (19) and (20).

Mr. Richardson's method consisted in taking a photograph of the smoke leaving a source and drifting down-wind. The exposure was not instantaneous, but extended over such a long period that a kind of composite photogragh was obtained showing the outer limits of the region containing the smoke. The general shape of the outline of this region is shown in Figs. 4 and 5; it is, as has been explained, a parabola with a pointed vertex. In some cases the paraboloidal part of the surface joined straight on to the conical part, as shown in Fig. 4, but in other cases there was a sort of neck between them as shown in Fig. 5. According to the theory set forth above this neck would be anticipated in cases where the R_ξ curve contained negative values as shown in Fig. 3. An R_ξ curve of this type might be due to some sort of regularity in the eddies of which the turbulent motion consists.

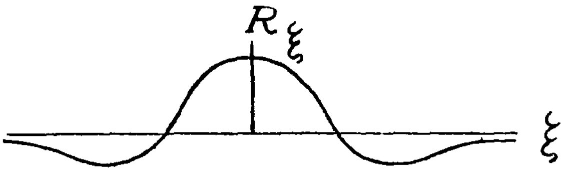


Fig. 3.

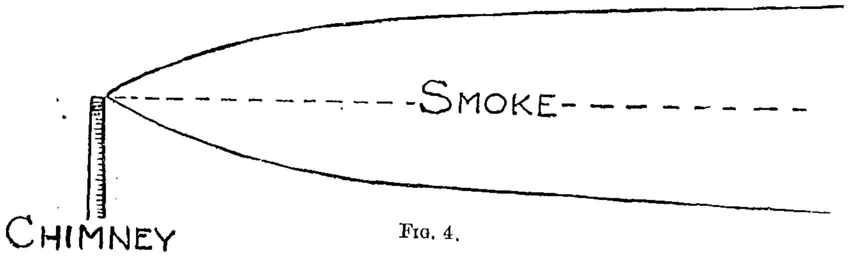


Fig. 4.

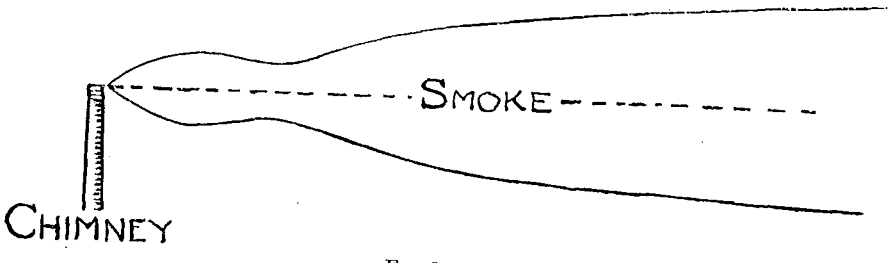


Fig. 5.

It appears that both theory and observation indicate that $[Xu]$ becomes constant after a certain interval of time (which depends of course on the value of ξ at which $\int_0^\xi R_\xi \, d\xi$ becomes practically constant with increasing values of ξ). This is a matter of considerable interest in the theory of the conduction of heat by means of turbulence, because it indicates a reason why the "diffusing power" of any type of turbulence appears to depend so little on the molecular conductivity and viscosity of the fluid.

After writing this paper I showed it to Mr. Richardson, who informed me that he had already noticed the relations (11), and at my request he sent me his proof which follows.

Note on a Theorem by Mr. G. I. Taylor on Curves which Oscillate Irregularly

By LEWIS F. RICHARDSON.

The theorem referred to is proved on the hypothesis that the standard deviations of p, dp/dt, d^2p/dt^2, ..., d^np/dt^n are constant over any long time. It also follows, as will now be shown, from the rather different hypotheses which may be stated thus:—

(i) No one of p, dp/dt, d^2p/dt^2 has a standard deviation less than a certain lower limit. (1)

(ii) The instantaneous values (t being time) of p, dp/dt, d^2p/dt^2, ..., never exceed in numerical value a certain upper limit. (2)

We might state simple numerical upper and lower limits. But as we are dealing with oscillations, it will be as well to take a hint from the properties of the sine curve. If $p = c \sin sp$, then $|d^np/dt^n|$ is not greater than cs^n, and the standard deviation of d^np/dt^n is $\sqrt{\tfrac{1}{2}}\, cs^n$.

For our irregular curve let us define B and r and A so that $|p| < B$ and $|d^np/dt^n| < Br^n$. (3)

The standard deviation of p is greater than A and that of d^np/dt^n is greater than Ar^n. (4)

It is required to find

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p \, \frac{d^{2n}p}{dt^{2n}} \, dt. \tag{5}$$

Integrate by parts, successively, so as to differentiate the p and to integrate $d^{2n}p/dt^{2n}$ until they both coincide in $d^n p/dt^n$. For example, when $n=5$, the result is

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{10}p}{dt^{10}}\,dt$$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[-\frac{d^9p}{dt^9}p+\frac{d^8p}{dt^8}\frac{dp}{dt}-\frac{d^7p}{dt^7}\frac{d^2p}{dt^2}+\frac{d^6p}{dt^6}\frac{d^3p}{dt^3}-\frac{d^5p}{dt^5}\frac{d^4p}{dt^4}\right]$$

$$+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left(\frac{d^5p}{dt^5}\right)^2 dt. \tag{6}$$

The expression in square brackets is less than $5B^2r^9$ however long the interval (t_2-t_1) may be, while $\int_{t_1}^{t_2}\left(\frac{d^5p}{dt^5}\right)^2 dt$ is greater than $(t_2-t_1)A^2r^{10}$, and so increases with the interval.

Thus when (t_2-t_1) is large enough, the term in square brackets becomes negligible. Generalizing the example, and taking account of the changes of sign introduced by partial integration,

$$\lim_{t_2-t_1\to\infty}\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{2n}p}{dt^{2n}}\,dt \text{ is } \frac{(-1)^n}{t_2-t_1}\int_{t_1}^{t_2}\left(\frac{d^np}{dt^n}\right)^2 dt. \tag{7}$$

If in place of $d^{2n}p/dt^{2n}$ in (5) we had had a coefficient of odd order, the partial integrations, when pursued so as to lead back again to the original form, would have produced an arrangement of signs such that like terms were added. So that

$$\lim_{t_2-t_1\to\infty}\frac{1}{t_2-t_1}\int_{t_1}^{t_2} p\,\frac{d^{2n+1}p}{dt^{2n+1}}\,dt = 0. \tag{8}$$

This depends on the hypothesis (2) only. Hypothesis (1) does not come in here. It was needed in proving (7).

Statistical Theory of Turbulence

By G. I. TAYLOR, F.R.S.

(*Received July* 4, 1935)

INTRODUCTION AND SUMMARY OF PARTS I–IV

Since the time of Osborne Reynolds it has been known that turbulence produces virtual mean stresses which are proportional to the coefficient of correlation between the components of turbulent velocity at a fixed point in two perpendicular directions. The significance of correlation between the velocity of a particle at one time and that of the same particle at a later time, or between simultaneous velocities at two fixed points was discussed in 1921 by the present writer in a theory of " Diffusion by Continuous Movements." The recent improvements in the technique of measuring turbulence have made it possible actually to measure some of the quantities envisaged in the theory and thus to verify some of the relationships then put forward.

The theory has also been developed in several directions which were not originally contemplated. The theory, as originally put forward, provided a method for defining the scale of turbulence when the motion is defined in the Lagrangian manner, and showed how this scale is related to diffusion. It is now shown that it can be applied either to the Lagrangian or to the Eulerian conceptions of fluid flow.

Where turbulence is produced in an air stream with a definite scale by means of a honeycomb or regular screen, either conception can be used to define a length which is related to certain measurable properties of flow and is a definite fraction of the mesh-length, M, of the turbulence-producing screen.

The Lagrangian conception leads to a length l_1, which is analogous to the " Mischungsweg " of Prandtl. Experiments on diffusion behind screens, Part IV, show that $l_1 = 0\cdot 1$ M. The Eulerian conception leads to a definite length l_2 which might be regarded as the average size of an eddy., Correlation measurements with a hot wire, Part II, show that l_2 is about equal to $0\cdot 2$ M.

The theory applied in the Eulerian manner to these correlation measurements also contains implicitly a definition of λ, " the average size of the smallest eddies," which are responsible for the dissipation of energy by viscosity.

It is proved that

$$\overline{W} = 15\mu\,(\overline{u^2}/\lambda^2,),$$

where $\overline{u^2}$ is the mean square variation in one component of velocity and $\overline{W}$ is the rate of dissipation of energy. This relationship is verified experimentally (Part II).

The relationship between λ and M is discussed and it is predicted that turbulence in an air stream moving with velocity U will die down so that

$$\frac{U}{\sqrt{\overline{u^2}}} = A\frac{x}{M} + B,$$

provided that the scale of turbulence is determined by the mesh-length M where A is a universal constant and B depends on the choice of the origin taken for x (the down-stream co-ordinate); u is a component of turbulent velocity. This theoretical relationship is compared with results of experiments carried out in wind tunnels in England and in America.

The theory is applied in Part III, to determine the distribution of dissipation across the section of a parallel wall channel (two-dimensional pipe) and it is shown that in the region near the walls turbulent energy is produced more rapidly than it is dissipated. In the central region the reverse is the case.

In Part IV the results of diffusion experiments made in America and at the National Physical Laboratory are discussed and it is shown that a complete set of such measurements can give $\sqrt{\overline{v^2}}$, l_1, and a length λ_η which may be regarded as a measure of the "smallest size of eddy" in the Lagrangian system. λ_η is connected, through the Lagrangian equations of motion, with the average spatial rate of change in pressure, namely

$$\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$$

by the formula

$$\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}} = \sqrt{2}\,\rho\frac{\overline{v^2}}{\lambda_\eta}.$$

Finally it is shown that the theory leads to the prediction that λ_η is a constant multiple of λ. The only set of experiments which exists at present gives $\lambda_\eta = 2\lambda$ approximately.

All the above results are subject to the restriction that the "Reynolds Number of Turbulence," namely $l\sqrt{\overline{u^2}}/\nu$, is greater than some number which must be determined by experiment.

PART I

At an early stage in the development of the theory of turbulence the idea arose that turbulent motion consists of eddies of more or less definite range of sizes. This conception combined with the already existing ideas of the Kinetic Theory of Gases led Prandtl and me independently to introduce the length l which is often called a " Mischungsweg " and is analogous to the " mean free path " of the Kinetic Theory. The length l could only be defined in relation to the definite but quite erroneous conception that lumps of air behave like molecules of a gas, preserving their identity till some definite point in their path, when they mix with their surroundings and attain the same velocity and other properties as the mean value of the corresponding property in the neighbourhood. Such a conception must evidently be regarded as a very rough representation of the true state of affairs. If we consider a number of particles or small volumes of fluid starting from some definite level and carrying, say, heat in a direction transverse to the mean stream lines, their average distance from the level at which they started will go on increasing indefinitely so that we can only consider a " Mischungsweg " in relation to some arbitrary time of flight during which we must consider that the particles preserve their individual properties distinct from those of their surroundings. Clearly this is an arbitrary conception and if pursued logically probably leads to a definitely wrong result. The only way in which a small volume can lose its heat is by conductivity to its surroundings. A decrease in molecular conductivity would therefore lead to an increasing time during which the small volume would retain its heat distinct from its surroundings and consequently a decrease in conductivity would necessarily lead to an increase in the " Mischungsweg." In all theories which make use of l it is assumed that l depends only on the *dynamical* conditions of the fluid and is nearly independent of such physical constants as thermal conductivity.

In all applications of " Mischungsweg " theories the length l is considered only in relation to further, more or less arbitrary, assumptions concerning the effect of turbulence on the mean motion or of the mean motion on turbulence. It appears as a fictitious length, the existence of which is detected only by observations of the distribution of mean velocity, temperature, etc.

The difficulty of defining a " Mischungsweg," or scale of turbulence, without recourse to some definite hypothetical physical process which bears no relation to reality does not arise in such applications. The difficulty, however, still exists and it led me, some years ago, to introduce the idea* that the scale of turbulence and its statistical properties in general can be given an exact interpretation by considering the correlation between the velocities at various points of the field at one instant of time or between the velocity of a particle at one instant of time and that of the same particle at some definite time, ξ, later. Some general relations applicable to either of these two aspects of the turbulent field were discussed, and the application of the definitions used in the second of them to diffusion in one dimension was worked out in detail. In this application of the theory the particles are conceived to move irregularly but with continuous velocity, v and $\overline{v^2}$ is supposed to be independent of time. The diffusion of particles starting from a point ($y = 0$) is shown to depend on the correlation R_ξ between the velocity of a particle at any instant and that of the same particle after an interval of time ξ. In continuous turbulent movements R_ξ must be a function of ξ such that $R_\xi = 1$ when $\xi = 0$ and $R_\xi \to 0$ when ξ is large.

If $\overline{Y^2}$ is the mean square of the distance through which the particles have diffused in time t it was proved that

$$\tfrac{1}{2}\frac{d}{dt}(\overline{Y^2}) = \overline{Yv} = \overline{v^2}\int_0^t R_\xi \, d\xi. \qquad (1)$$

If the time of diffusion is small so that R_ξ has not departed appreciably from its initial value 1·0, (1) becomes

$$\tfrac{1}{2}\frac{d}{dt}(\overline{Y^2}) = \overline{v^2}t$$

so that

$$\sqrt{\overline{Y^2}} = v't, \qquad (2)$$

where $v' = \sqrt{\overline{v^2}}$.

If the diffusion is taking place in a stream of air moving with velocity U and if the spread is observed at a small distance x down-stream from the source $t = x/U$ so that

$$\frac{\sqrt{\overline{Y^2}}}{x} = \frac{v'}{U}. \qquad (3)$$

If the irregular motion is of such a character that it is possible to define a time T such that $R_\xi = 0$ for all values of ξ greater than T, so that there is no correlation between the velocities of a particle at the beginning and end of the time interval T, then

$$\overline{Yv} = \overline{v^2}\int_0^T R_\xi d_\xi, \qquad (4)$$

* 'Proc. Lond. Math. Soc.,' vol. 20, p. 196 (1921).

$\overline{Yv}$ is therefore constant for all values of $t > T$ in spite of the fact that the value of $\overline{Y^2}$ is continually increasing and $\overline{v^2}$ is constant.

Under these circumstances it is possible to define a length l_1, such that

$$l_1 \sqrt{\overline{v^2}} = \overline{v^2} \int_0^T R_\xi \, d_\xi = \tfrac{1}{2} \frac{d}{dt} (\overline{Y^2}). \tag{5}$$

It will be seen from (5) that the length l_1, defined as

$$l_1 = \sqrt{\overline{v^2}} \int_0^T R_\xi \, d_\xi , \tag{6}$$

bears the same relationship to diffusion by turbulent motion that the mean free path does to molecular diffusion. In this sense it is very similar to the "Mischungsweg," l, but with this important difference that the question of mixture does not arise in defining it.

As is pointed out above, theories which depend essentially on the idea of mixture by subdivision and ultimate molecular diffusion lead to the expectation that the "Mischungsweg" will depend very greatly on the molecular diffusive power of the fluid. In the theory of diffusion by continuous movements the length l_1 bears no relation to any process of mixture, indeed it is equally valid if mixture never takes place. The effect of molecular diffusion would be to prevent the fluid from becoming ever increasingly "spotty," *i.e.*, it would tend to prevent a continual increase in the deviations of the measurable properties of the fluid from their mean value in the neighbourhood. Mixture has no effect in this theory on the diffusive power of turbulent motion.

Correlation in the Turbulent Field when described in the Eulerian Manner

In a loose way it has been thought that the "Mischungsweg" length l is related to, and even may be taken as a measure of the average size of the larger eddies in turbulent flow. It will be noticed that in the original "Mischungsweg" theories, and also in the theory of diffusion by continuous movements, everything is defined in a Lagrangian manner, *i.e.*, by following the paths of particles. When a field of eddying flow is considered as an entity in itself, apart from its effect as a diffusive agent, it is more usual to think in terms of the Eulerian conception of fluid flow, *i.e.*, a field of stream lines conceived to exist in space at one instant of time. Any ideas we may have about "the size of an eddy" are likely to be formulated in the Eulerian system. For this reason it would not be possible to connect directly the size of an eddy, even if it could be accurately defined, with the value of l or of l_1 as defined by (6) in the Lagrangian system. At the same time it seems to be a matter of considerable theoretical interest to investigate the statistical properties of a field of turbulent flow when described in the Eulerian manner, with a view to defining a length which may represent in some definite way the " size of an eddy."

The correlation theory developed in my paper, " Diffusion by Continuous Movements," is equally applicable in this case and may be used to formulate another definition of the scale of turbulence. It is clear that whatever we may mean by the diameter of an eddy a high degree of correlation must exist between the velocities at two points which are close together when compared with this diameter. On the other hand, the correlation is likely to be small between the velocity at two points situated many eddy diameters apart. If, therefore, we imagine that the correlation R_y between the values of u at two points distant y apart in the direction of the y co-ordinate has been determined for various values of y we may plot a curve of R_y against y, and this curve will represent, from the statistical point of view, the distribution of u along the y axis. If R_y falls to zero at, say, $y = Y$, then a length l_2 can be defined such that

$$l_2 = \int_0^\infty R_y\,dy = \int_0^Y R_y\,dy. \tag{7}$$

This length l_2 may be regarded as the analogue in the Eulerian system of l_1, which is defined in the Lagrangian system. It may be taken as a possible definition of the " average size of the eddies."

Experimental Methods for Measuring l_1 and l_2

The compensated hot wire is capable of being used to measure several of the quantities which are necessarily considered in any statistical theory of turbulence.

(1) $\overline{u^2}$ can be measured by means of a hot wire anemometer. If the amplified disturbances are passed through a wire the heat produced can give rise to a current in a thermojunction, which will cause a deflection in a galvanometer proportional to $\overline{u^2}$.

(2) If two hot wires are set up at a distance y apart transverse to a stream of air and the currents produced by variations in u at the two points are sent through the two coils of an electric dynamometer, the resulting deflection will be proportional to $\overline{u_0 u_y}$ where u_0 and u_y are the

velocities at the two points. In this way $R_y = \overline{u_0 u_y}/\overline{u^2}$ can be measured. By repeating these measurements for a number of different distances of separation y between the two hot wires, R_y can be determined for all values of y and hence by integration l_2 can be found. The (R_y, y) curve has already been obtained in certain cases by Messrs. Simmons and Salter at the National Physical Laboratory by this method (*see* fig. 1 of Part II).

Another method is to arrange two equal hot wires on two arms of a Wheatstone bridge thus measuring $\overline{(u_1 - u_0)^2}$. If $\overline{u_0^2}$ and $\overline{u_1^2}$ are measured independently at the two stations, $\overline{u_0 u_1}$ can be found from the relationship

$$\overline{u_0^2} + \overline{u_1^2} - \overline{(u_0 - u_1)^2} = 2\overline{u_0 u_1}. \tag{8}$$

Yet another method due to Prandtl* is to pass the currents from the two hot wires through coils which cause deflections of a spot of light in two directions at right angles to one another. If the two hot wires are identical and so close together that the correlation is nearly 1·0, the spot of light moves over a very elongated elliptic area, the long axis of which is at 45°, to the deflections caused by either of the wires in the absence of disturbances from the other. By measuring the ratio of the principal axes of the elliptical blackened areas produced on a photographic plate by the moving spot of light during a prolonged exposure, it is possible to calculate R_y. This method is specially suitable for measurements when the correlation is very high, *i.e.*, $1 - R_y$ is small. It is not so suitable for small correlations as the electric dynamometer method. Correlation measurements made in this way are shown in fig. 1 of Part III of this paper.

(3) By introducing heat at a concentrated source or a line source in an air stream and measuring the spreading of the heat to leeward of the source it should be possible to measure the quantity $\frac{d}{dt}\overline{Y^2}$ which occurs in (1) and hence to find $\int_0^t R_\xi \, d_\xi$ for various values of t. If this reaches a constant value at some distance down-stream then l_1 can be found. This method was suggested in my paper on "Diffusion by Continuous Movements." Up to the present, however, the theory has only been applied to cases like that of diffusion in the atmosphere† where there is no *a priori* reason to suppose that any definite scale of turbulence can be

* Prandtl and Reichardt, "Einfluss von Wärmeschichtung auf die Eigenschaften einer Turbulenter Strömung." Deutsche Forschung, p. 110 (1934).

† Sutton, 'Proc. Roy. Soc.,' A, vol. 135, p. 143 (1932).

defined. Indeed, Mr. O. G. Sutton has shown that the best representation of diffusion in the air near the ground is obtained by assuming $R_\xi \propto \xi^{-n}$ so that R_ξ does not vanish however great ξ may be. In fact $\int_0^t R_\xi \, d_\xi$ increased continuously with increase in t so that l_1, defined as in equation (6), would have no definite value.

The turbulence which occurs in wind tunnels is produced or controlled by a honeycomb with cells of a definite size. In a wind tunnel, therefore, there is an *a priori* reason why the turbulence might be expected to be of some definite scale. In fact, it might be expected that both l_1 and l_2 would be some definite fraction of the mesh of the cells. Under these circumstances the diffusion equations (1) and (6) reduce to

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = l_1 v'. \tag{9}$$

This expression is valid when the distance x of the points at which measurements of $\overline{Y^2}$ are made from the point or line source of diffusion is so great that $R_\xi = 0$ where $\xi = x/U$ and U is the mean speed of the air stream.

Application of Diffusion Equation when Turbulence is Decaying

In the air stream behind a grid or honeycomb the turbulence is not constant. It decreases as the distance down-stream increases. The preceding theory cannot then be applied without further investigation.

If $\overline{v^2}$ is considered as a function of t the diffusion equation is

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = \overline{v_t \int_0^t v_{t-\xi}\, d\xi} \tag{10}$$

for $Y = \int_0^t v_{t-\xi}\, d\xi$ and $\frac{d}{dt}(\overline{Y^2})$ is the rate of increase in $\overline{Y^2}$ at time t after the beginning of the diffusion from a concentrated source.

If ${}_tR_{t-\xi}$ is the coefficient of correlation between the velocity at time t and that at time $t - \xi$, (10) may be written

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = v'_t \int_0^t v'_{t-\xi}\,({}_tR_{t-\xi})\, d\xi, \tag{11}$$

where v'_t, $v'_{t-\xi}$ are written for $\sqrt{\overline{v^2_t}}$, $\sqrt{\overline{v^2_{t-\xi}}}$.

When the average condition of the turbulent motion is constant with respect to time ${}_tR_{t-\xi}$ is the same as ${}_tR_{t+\xi}$ or R_ξ and is a function of ξ

only, so that (11) is identical with (1). When v' is not constant, it is not possible to proceed beyond (11), but the existing experimental evidence seems to show that turbulent diffusion is proportional to the speed, so that if matter from a concentrated source is diffused over an area down-stream from the source, an increase in the speed of the whole system (*i.e.*, proportional increases in turbulent and mean speed) leaves the distribution of matter in space unchanged (though the absolute concentration is reduced). The condition that this may be so is that ${}_{t}R_{t-\xi}$ is a function of η only where

$$d\eta = v'\,d\xi = (v'/U)\,dx \tag{12}$$

and $x = Ut$ is the distance down-stream from the source.

The equation which represents the lateral spread of matter or heat from a concentrated source is therefore

$$\tfrac{1}{2}\frac{U}{v'}\frac{d}{dx}(\overline{Y^2}) = \int_0^{\eta_x} R_\eta\,d\eta, \tag{13}$$

where

$$\eta_x = \int_0^x \frac{v'}{U}\,dx, \tag{14}$$

and R_η is the correlation between the velocities of a particle at times t_1 and t_2 when $\eta = \int_{t_1}^{t_2} v'\,dt$. If R_η falls to zero at a finite value of η, say $\eta = \eta_1$, and remains zero for all greater values of η, $\int_0^{\eta} R_\eta\,d\eta$ is finite. If l_1 be written for $\int_0^{\eta_1} R_\eta d\eta$ then (11) becomes

$$\tfrac{1}{2}\frac{U}{v'}\frac{d}{dx}(\overline{Y^2}) = l_1. \tag{15}$$

This is the same expression as that found for turbulence which is not decaying.*

It is worth noticing that (13) may be expressed in the form

$$\tfrac{1}{2}\frac{d}{d\eta}(\overline{Y^2}) = \int_0^{\eta_x} R_\eta\,d\eta. \tag{16}$$

When η_x is small so that $R_\eta = 1$ over the range from 0 to x, (16) becomes

$$\tfrac{1}{2}\frac{d}{d\eta}(\overline{Y^2}) = \eta. \tag{17}$$

The integral of (17) is

$$\overline{Y^2} = \eta^2 \quad \text{or} \quad \sqrt{\overline{Y^2}} = \eta. \tag{18}$$

* *See* equations (1) and (6).

When the turbulence is constant $\eta = xv'/U$ so that (18) reduces to the previous expression (3) for the spread of matter near a concentrated source. If the turbulence is not constant and if $\overline{Y^2}$ and v'/U are measured at a number of values of x, then both η and $\frac{1}{2}\frac{U}{v'}\frac{d}{dx}(\overline{Y^2})$ can be found. Thus $\int_0^\eta R_\eta \, d\eta$ can be plotted against η and R_η can be found graphically from this experimental curve.

Micro-Turbulence and Dissipation of Energy

Besides the motions which are chiefly responsible for the diffusive power of turbulence the whole field may be in a state of micro-turbulence, *i.e.*, there may exist very small-scale eddies which, though they play a very small part in diffusion, yet may be the principal agents in the dissipation of energy. They may also be the principal causes of the effects of turbulence on the boundary layer in wind tunnel work because the absolute magnitude of the space rates of change in pressure may depend on them.

Dissipation of Energy

The rate of dissipation of energy in a fluid at any instant depends only on the viscosity, μ, and on the instantaneous distribution of velocity. If, therefore, the representation of the essential statistical properties of the velocity field can be expressed by the R_y curve and similar correlation curves it must be possible to deduce from them the rate of dissipation of energy. This would in general involve a complicated analysis, but the problem can be much simplified if the field of turbulent flow is assumed to be isotropic.

Isotropic Turbulence

In isotropic turbulence the average value of any function of the velocity components, defined in relation to a given set of axes, is unaltered if the axes of reference are rotated in any manner. That there is a strong tendency to isotropy in turbulent motion has long been known. It has been shown by Fage and Townend,* for instance, that the average values of the three components of velocity in the central region of a pipe of square section are nearly equal to one another. In the atmosphere the same phenomenon has been observed; though, as might be expected, the

* Townend, 'Proc. Roy. Soc.,' A, vol. 145 (1934) (*see* fig. 15, p. 203).

vertical components are smaller near the ground than the horizontal ones, this inequality decreases with height above the ground.*

The assumption of isotropy immediately introduces many simplifications both into the statistical representation of turbulence and into the expression for the mean rate of dissipation of energy.

The general expression for the rate of dissipation is

$$W = \mu \left\{ 2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} + \overline{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2} + \overline{\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2} + \overline{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2} \right\}. \quad (19)$$

Making the assumption that the turbulence is statistically isotropic, the relations

$$\left.\begin{array}{l} \overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} = \overline{\left(\frac{\partial w}{\partial z}\right)^2} \\ \text{and} \\ \overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial u}{\partial z}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial z}\right)^2} = \overline{\left(\frac{\partial w}{\partial x}\right)^2} = \overline{\left(\frac{\partial w}{\partial y}\right)^2} \\ \text{and} \\ \overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} = \overline{\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}} = \overline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}} \end{array}\right\} \quad (20)$$

are immediately obtained so that

$$\frac{W}{\mu} = 6\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 6\overline{\left(\frac{\partial u}{\partial y}\right)^2} + 6\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}. \quad (21)$$

Equation (21) contains three types of term. It will now be shown that these are all related to one another so that if the value of one is known the other two are known.

That relationships can be found between the mean values of squares and products of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, .., etc., is obvious. The simplest relationship is obtained as follows. The condition of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

so that

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} + \overline{\left(\frac{\partial v}{\partial y}\right)^2} + \overline{\left(\frac{\partial w}{\partial z}\right)^2} = -2\left(\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} + \overline{\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}} + \overline{\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}}\right). \quad (22)$$

The conditions of statistical isotropy therefore lead to the relationship

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = -2\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} \quad (23)$$

* Taylor, 'Q. J. R. Met. Soc.,' vol. 53, p. 210 (1927).

or

$$\frac{\overline{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}}}{\sqrt{\overline{\left(\dfrac{\partial u}{\partial x}\right)^2}}\sqrt{\overline{\left(\dfrac{\partial v}{\partial y}\right)^2}}} = -\tfrac{1}{2}. \tag{24}$$

In other words there is a definite correlation coefficient between $\partial u/\partial x$ and $\partial v/\partial y$ equal to $-\frac{1}{2}$.

Mean Value of General Quadratic Function of $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, ..., etc.

Consider the most general possible expression for the mean value of any quadratic function of the nine quantities

$$\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x},\ \frac{\partial w}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial y},\ \frac{\partial w}{\partial y},\ \frac{\partial u}{\partial z},\ \frac{\partial v}{\partial z},\ \frac{\partial w}{\partial z}.$$

In general there are 36 possible combinations of 9 things taken 2 at a time. Thus the most general quadratic expression contains 45 terms, namely the 9 squares of the quantities concerned and the 36 combinations of 2.

When the motion is statistically istotropic the 45 terms fall into 10 groups, each of which contains 3 or 6 means which are equal to one another; for example, one group containing 3 equal terms consists of

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2},\ \overline{\left(\frac{\partial v}{\partial y}\right)^2},\ \text{and}\ \overline{\left(\frac{\partial w}{\partial z}\right)^2}.$$

Another containing 6 equal terms consists of

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}},\ \overline{\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}},\ \overline{\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}},\ \overline{\frac{\partial v}{\partial y}\frac{\partial w}{\partial x}},\ \overline{\frac{\partial w}{\partial z}\frac{\partial u}{\partial y}},\ \overline{\frac{\partial w}{\partial z}\frac{\partial v}{\partial x}}.$$

The 10 possible independent mean values will be denoted by a_1, a_2, ..., a_{10} according to the scheme laid out in Table I where the top row of the table gives the type term and all other terms of the same type can be obtained by permuting symmetrically the elements of the type term.

The symbol which represents the mean value of any term of a type is given in the second row and the number of independent terms in each group is given in the last row.

In terms of these symbols (21) becomes

$$W/\mu = 6a_1 + 6a_3 + 6a_8. \tag{25}$$

TABLE I

Type	$\overline{\left(\frac{\partial u}{\partial x}\right)^2}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}$	$\overline{\left(\frac{\partial u}{\partial y}\right)^2}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$	Total
Symbol	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
Number of terms in group	3	6	6	3	6	3	6	3	6	3	45

I now propose to prove that the 10 values, $a_1, a_2, \ldots a_{10}$, are interconnected, so that if the value of any one of them, which is not zero, is known all the rest are known. For this purpose it is necessary to prove 9 linear relationships. One such relationship has already been proved (*see* equation (23)). Expressed in the symbols of Table I (23) may be written

$$a_1 = -2a_6. \tag{26}$$

Further relationships may be obtained as follows. Take any one of the 45 possible terms in the most general quadratic expression involving the 9 partial differentials of (25). Transform u, v, w, x, y, z, by rotation of the axes to u', v', w', x', y', z'. The transformed expression will still be quadratic but will contain terms of other types than the original one. When the mean values of the terms in the transformed expression are considered it is a necessary consequence of the definition of isotropy that the value of each is equal to that of the type term in the group in which they are classed. A simple transformation is obtained by rotating the axes through 45° about the axis of z so that

$$\left.\begin{aligned} \sqrt{2}\,x' &= x + y \\ \sqrt{2}\,y' &= -x + y \\ z' &= z \end{aligned}\right\} \quad \left.\begin{aligned} \sqrt{2}\,u' &= u + v \\ \sqrt{2}\,v' &= -u + v \\ w' &= w \end{aligned}\right\} \tag{27}$$

Hence

$$\begin{aligned} \frac{\partial u}{\partial x} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) & \frac{\partial u}{\partial y} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial v}{\partial x} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) & \frac{\partial v}{\partial y} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial w}{\partial x} &= \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} - \frac{\partial w'}{\partial y'}\right) & \frac{\partial w}{\partial y} &= \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} + \frac{\partial w'}{\partial y'}\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} - \frac{\partial v'}{\partial z'}\right) \\ \frac{\partial v}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial z'}\right) \\ \frac{\partial w}{\partial z} &= \frac{\partial w'}{\partial z'} \end{aligned} \tag{28}$$

Take, for example, $\overline{\left(\frac{\partial u}{\partial x}\right)^2} = a_1$. By squaring the transformed expression for $\left(\frac{\partial u}{\partial x}\right)$, taking the mean value and substituting the symbol for the corresponding type term from Table I it will be found that

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = a_1 = \tfrac{1}{4}\left\{\overline{\left(\frac{\partial u'}{\partial x'}\right)^2} + \overline{\left(\frac{\partial v'}{\partial x'}\right)^2} + \dots \overline{\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'}} \dots\right\}$$

$$= \tfrac{1}{2}(a_1 + a_3 - a_5 - a_2 + a_6 + a_8 - a_2 - a_5). \quad (29)$$

Similarly

$$\overline{\left(\frac{\partial v}{\partial y}\right)^2} = a_1 = \tfrac{1}{2}(a_1 + a_3 + a_5 + a_2 + a_6 + a_8 + a_2 + a_5), \quad (30)$$

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = a_3 = \tfrac{1}{2}(a_1 + a_3 - a_5 + a_2 - a_6 - a_8 + a_2 - a_5), \quad (31)$$

$$\overline{\left(\frac{\partial v}{\partial x}\right)^2} = a_3 = \tfrac{1}{2}(a_1 + a_3 + a_5 - a_2 - a_6 - a_8 - a_2 + a_5). \quad (32)$$

From these equations it will be found that

$$a_2 = a_5 = 0 \quad (33)$$

and

$$a_1 - a_3 - a_6 - a_8 = 0. \quad (34)$$

No further relations can be derived by transforming the type terms corresponding with a_2, a_5, a_6, or a_8. Proceeding to terms involving w or z

$$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}} = a_{10} = \tfrac{1}{2}\left\{\overline{\left(\frac{\partial u'}{\partial z'}\right)^2} - \overline{\left(\frac{\partial v'}{\partial z'}\right)^2}\right\} = \tfrac{1}{2}(a_3 - a_3) = 0, \quad (35)$$

$$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = a_9 = \frac{1}{2\sqrt{2}}(a_2 - a_9 + a_4 - a_7 + a_7 - a_4 + a_9 - a_2) = 0, \quad (36)$$

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = a_7 = \frac{1}{2\sqrt{2}}(a_2 - a_9 - a_4 + a_7 + a_7 - a_4 - a_9 + a_2). \quad (37)$$

hence since

$$a_2 = a_9 = 0 \qquad a_7(\sqrt{2} - 1) + a_4 = 0 \quad (38)$$

$$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}} = a_4 = \frac{1}{2\sqrt{2}}(a_2 - a_9 + a_4 - a_7 - a_7 + a_4 - a_9 + a_2),$$

and hence

$$a_4(\sqrt{2} - 1) + a_7 = 0 \quad (39)$$

combining (38) with (39)

$$a_4 = a_7 = 0. \quad (40)$$

Summing up the results so far obtained 6 of the 10 independent types of mean are zero, namely, a_2, a_4, a_5, a_7, a_9, a_{10} and there are two independent relationships between the remaining 4 means, namely,

$$a_1 = -2a_6 \quad \text{and} \quad a_1 - a_3 - a_6 - a_8 = 0.$$

Of these the first depends on incompressibility and isotropy. The second depends only on isotropy.

One further relationship can be obtained by volume integration of the general dissipation expression (19). This integration is well known*: it is

$$\iiint \frac{W}{\mu}\,\partial x\,\partial y\,\partial z = \iiint \left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\} dx\,dy\,dz$$

$$= \iiint (\xi^2 + \eta^2 + \zeta^2)\,dx\,dy\,dz - \iint \frac{\partial}{\partial n}(q^2)\,dS + 2\iint \begin{vmatrix} l & m & n \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix} dS, \tag{41}$$

where

$$\xi = (\partial w/\partial y) - (\partial v/\partial z), \text{ etc.}$$

and the integrals are taken over the cloud surface S and through its volume. If the closed surface is large compared with the scale of the turbulence the surface integrals are small compared with the volume integrals which may therefore be neglected. Taking the mean value of all the quantities in (41) and expressing the result for isotropic turbulence in terms of the symbols of Table I, (41) becomes

$$\overline{W}/\mu = 6a_1 + 6a_3 + 6a_8 = 6a_3 - 6a_8. \tag{42}$$

Hence

$$a_1 + 2a_8 = 0, \tag{43}$$

solving (26), (34), and (43) it will be seen that

$$a_1 = \tfrac{1}{2}a_3 = -2a_6 = -2a_8. \tag{44}$$

Three obvious corollaries to this result may be noticed:

(1) The correlation coefficient between $\partial u/\partial x$ and $\partial v/\partial y$ is $-\frac{1}{2}$.

(2) The correlation coefficient between $\partial u/\partial y$ and $\partial v/\partial x$ is $-\frac{1}{4}$.

(3) When the mean value of any one of the four possible types of quadratic terms which are not zero is known all the rest are known, so that the mean value of any quadratic function of the space rates of change

* *See*, for instance, the chapter on viscosity in Lamb's " Hydrodynamics."

of velocity is also known. In particular the dissipation may be expressed in terms of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$. The correct expression is

$$\overline{W}/\mu = 6a_1 + 6a_3 + 6a_8 = 3a_3 + 6a_3 - 1\cdot 5a_3 = 7\cdot 5\,\overline{\left(\frac{\partial u}{\partial y}\right)^2}. \tag{45}$$

Statistical Representation of Microturbulence

The value of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ is clearly related to the way in which the value of R_y falls off from its initial value $1\cdot 0$ as y increases from zero. I have proved,* in fact, that

$$R_y = 1 - \frac{1}{2!}\frac{y^2}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial y}\right)^2} + \frac{1}{4!}\frac{y^4}{\overline{u^2}}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2} - \dots \tag{46}$$

The curvature of the R_y curve at $y = 0$ is therefore a measure of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ so that

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = 2\overline{u^2} \operatorname*{Lt}_{y\to 0}\left(\frac{1 - R_y}{y^2}\right). \tag{47}$$

The significance of the expression (47) can best be appreciated by defining a length λ such that

$$\frac{1}{\lambda^2} = \operatorname*{Lt}_{y\to 0}\left(\frac{1 - R_y}{y^2}\right), \tag{48}$$

λ^2 is then a measure of the radius of curvature of the R_y curve at $y = 0$. If the curve is drawn on such a scale that its height is H (corresponding with $R_y = 1$ at $y = 0$) the radius of curvature at $y = 0$ is $\lambda^2/2H$.

Another interpretation of λ may be found by describing the parabola which touches the R_y curve at the origin. This parabola will cut the axis $R_y = 0$ at the point $y = \lambda$. λ may roughly be regarded as a measure of the diameters of the *smallest* eddies which are responsible for the dissipation of energy.

Connection between Dissipation of Energy and Correlation Function R_y

Combining (45) with (47) and (48), the dissipation is related to the correlation function R_y by the equation

$$\overline{W} = 15\,\mu\overline{u^2} \operatorname*{Lt}_{y\to} \frac{1 - R_y}{y^2} \tag{49}$$

* 'Proc. Lond. Math. Soc.,' vol. 20, p. 205, equation (14), (1921).

or

$$\overline{W} = 15\mu\overline{u^2}/\lambda^2. \tag{50}$$

Since $\overline{u^2}$ and R_y can be measured directly by means of the hot wire technique referred to earlier, the relationship (49) can be verified if $\overline{W}$ can be measured by other means. The way in which this can be done and the comparison between this statistical theory and the results of observation will be discussed later. In the meantime it may be noticed that if the Reynolds's stresses in geometrically similar fields of flow are proportional to $\overline{u^2}$ or u'^2, $\overline{W}$ is proportional to u'^3, so that λ is proportional to $(u')^{-\frac{1}{2}}$, and since λ is proportional to the curvature of the R_y curve at $y = 0$ we are led to the prediction that the curvature of the R_y curve at its summit, $y = 0$, will be proportional to $1/u'$. In the limit for very high values of u' the R_y curve may be expected to have a *pointed* top.

Suggestion for Experimental Test in Wind Tunnel of Predicted Correlation Relations

It has been shown how measurements of correlation between the readings of two hot wires at points close together in a transverse section of a pipe or wind tunnel can give the value of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$. If similar measurements could be made in a line parallel to the main stream, values of $\overline{\left(\frac{\partial u}{\partial x}\right)^2}$ could be obtained in the same way. Equation (44) shows that

$$a_1 = \overline{\left(\frac{\partial u}{\partial x}\right)^2} = \tfrac{1}{2}\, a_3 = \tfrac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}$$

and referring to equation (47) which is equally true when x is substituted for y, it will be seen that for the correlation to fall a given amount from its coincidence value $1 \cdot 0$ the separation of the two hot wires must be $\sqrt{2}$ times as great when one lies up- or down-stream from the other as it is when they lie across the stream.

This is a definite new theoretical prediction which could be tested. If difficulty is found in working with one hot wire down-stream from the other, measurements might be made with the two wires mounted at a fixed distance r apart on a rotating holder, and the variation in the correlation R as the holder is rotated might be found.

The correlation between the values of u observed at two points situated at a short distance, r, apart in a line making an angle θ to the wind direction is*

$$R = 1 - \frac{r^2}{2\overline{u^2}}\overline{\left(\frac{\partial u}{\partial x'}\right)^2}, \tag{51}$$

* Compare equation (47).

where $x' = x \cos \theta + y \sin \theta$. Since

$$\frac{\partial u}{\partial x'} = \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y}$$

(51) becomes, in the notation of Table I,

$$R = 1 - \frac{r^2}{2\overline{u^2}}(a_1 \cos^2 \theta + a_3 \sin^2 \theta + 2a_2 \cos \theta \sin \theta). \qquad (52)$$

When the turbulence is isotropic this is

$$1 - R = \frac{r^2}{2\overline{u^2}}\overline{\left(\frac{\partial u}{\partial y}\right)^2}(\cos^2 \theta + \tfrac{1}{2} \sin^2 \theta),$$

hence from the definition† of λ

$$1 - R = \frac{r^2}{\lambda^2}(\cos^2 \theta + \tfrac{1}{2} \sin^2 \theta). \qquad (53)$$

It appears, therefore, that $1 - R$ should vary in the ratio 2: 1 as the holder is rotated for the maximum to the position to maximum to minimum correlation.

Dimensional Relationship between λ and Scale of Turbulence

It has been shown by v. Karman that if the surface stress in a pipe is expressed in the form $\tau = \rho v_{\times}^2$ then

$$\frac{U_0 - U}{v_{\times}} = f\left(\frac{r}{a}\right), \qquad (54)$$

where U_0 is the maximum velocity in the middle of the pipe and U is the velocity at radius r. This relationship is associated with the conception that the Reynolds's stresses are proportional to the squares of the turbulent components of velocity. It seems that the rate of dissipation of energy in such a system must be proportional, so far as changes in linear dimensions, velocity, and density are concerned, to $\rho u'^3/l$, where l is some linear dimension defining the scale of the system. For turbulence produced by geometrically similar boundaries therefore

$$W = \text{constant}\left(\frac{\rho u'^3}{l}\right) = 15 \frac{\mu u'^2}{\lambda^2}.$$

For such systems therefore

$$\frac{\lambda^2}{l^2} = C \frac{\nu}{lu'}, \qquad (55)$$

† *See* equation (48).

where C depends on the position relative to the solid boundaries of the point at which observations are made and on the element used for defining *l*.

Application to Air Stream Behind Regular Grids or Honeycombs

Formula (55) is specially well adapted for discussing the decay of turbulence in an air stream behind a grid or honeycomb, because it has been found that at a certain distance down-stream the stream becomes statistically uniform, *i.e.*, the " wind shadow " of the grid disappears and the mean velocity becomes uniform. Under these circumstances it seems that the C of formula (55) must be a constant for any definite form of grid. The researches of Schlichting* have shown that at a short distance behind a cylindrical obstacle the wake assumes a definite form. The width of the wake and the velocity of the air in the middle of the wake depend on the drag coefficient of the obstacle so that obstacles of very varied cross-sections produce identical wakes provided their drag coefficients are identical. For this reason it may be expected that if a regular grid or honeycomb is constructed the scale of the turbulent motion produced by it at any distance down-stream beyond the point where the " wind-shadow " has disappeared will depend only on the form and mesh size of the grid, and not on the cross-section of the bars or sheets from which it is constructed. On the other hand, the velocities of the turbulent components will certainly depend on the drag coefficient of the bars themselves as well as on the distance down-stream from the grid at which measurements are made.

These considerations lead to the prediction that if only one form of mesh is considered, say a square mesh, and if the length *l* in (55) is taken as M, the mesh length, *i.e.*, the side of each square of the mesh, then the constant C in (55) will be an absolute constant independent of the form of the bars of the grid. We are thus led to a definite expression for λ/M namely,

$$\frac{\lambda}{M} = A \sqrt{\frac{\nu}{Mu'}}, \qquad (56)$$

where A is an absolute constant for all grids of a definite type, *e.g.*, for all square-mesh grids or honeycombs.

* ' Ingen. Arch.,' vol. 1, p. 533 (1930).

Prediction of Law of Decay of Turbulence Behind Grids and Honeycombs

We are now in a position to predict the way in which turbulence may be expected to decay when a definite scale has been given to it as the air stream passes through a regular grid or honeycomb.

The rate of loss of kinetic energy of the turbulence per unit volume is

$$-\tfrac{1}{2}\rho U \frac{d}{dx}(\overline{u^2}+\overline{v^2}+\overline{w^2}),$$

which in an isotropic field of turbulence is

$$-\tfrac{3}{2}\rho U \frac{d}{dx}(\overline{u^2}).$$

This must be equal to the rate of dissipation W, so that

$$-\tfrac{3}{2}\rho U \frac{d}{dx}(\overline{u^2}) = 15\mu \frac{\overline{u^2}}{\lambda^2}. \qquad (57)$$

This equation ,is capable of experimental verifications independently of the relationship (56) between λ and M because, as has been shown, λ is connected with R_ν through (48) and R_ν can be measured instrumentally.

On the other hand, if the relationship (56) between λ and M is assumed to hold it is possible to calculate the law of decay of turbulence. Substituting for λ from (56), (57) becomes

$$-\frac{U}{u'^3}\frac{d}{dx}(u'^2) = \frac{10}{MA^2}, \qquad (58)$$

and integrating (58) the following very simple law of decay is predicted,

$$\frac{U}{u'} = \frac{5x}{A^2M} + \text{constant}. \qquad (59)$$

This expression should be applicable to all cases where the turbulence is of a definite scale. The linear law of increase in U/u' should therefore apply to all wind tunnels where the scale of turbulence is controlled by a honeycomb or grid, and the value of the constant A determined experimentally, using (59), should be universal for all square grids. Thus, the turbulence behind a square-section honeycomb with long cells should obey the same law of decay as that produced by a square-mesh grid of flat slats or a square-mesh grid of round bars, and the values of A should be identical in all these cases.

For other types of grid or honeycomb, *e.g.*, with hexagonal or triangular cells or a grid of parallel slats or plates, the constant A determined experimentally by applying (59) to observed values of u' at different distances down the air stream might be expected to assume other values.

Expected Limitations to Predicted Linear Law of Decay of Turbulence

This is a very comprehensive prediction, but it is subject to certain limitations. In the first place it cannot be expected to apply when Mu'/ν is small, for equations of the type (56) are not true when Mu'/ν is small. In fact if (56) were supposed to hold when Mu/v is small λ would be greater than M, a condition which is clearly impossible at any rate near the grid.

A second restriction is that the formula cannot be expected to apply in the region immediately behind the grid where the mean velocity is variable, *i.e.*, where the " shadow " of the grid is still distinct. It is found experimentally that when the diameter of the bars of the grid is small compared with M the shadow may extend to as much as 20 M or 30 M behind the grid, but when the bars are as broad as $\frac{1}{4}$ M the shadow disappears a few mesh lengths down-stream from the grid.

A third limitation may be expected to operate when the turbulence is not entirely due to the grid through which the stream passes. If, for instance, a very turbulent stream passes through a grid consisting of thin wires arranged in a large-scale mesh the scale of the turbulence in the stream might hardly be affected by its passage through the grid.

Summary of Results and Theoretical Predictions

(1) When the turbulence of a definite scale is produced or controlled in a stream of air by a honeycomb or grid of regularly spaced bars the scale of turbulence can be investigated in two ways. If the Lagrangian conception of fluid motion is adopted the scale of turbulence can be defined in reference to the correlation R_ξ between the velocity of a particle and that of the same particle at time ξ later. This conception is suited for discussing experiments on diffusion of heat from a concentrated source.

(2) If the diffusive spread of heat or matter from a line source is measured near the source it is proportional to the distance from the source and measures the transverse component of turbulent velocity independently of the scale.

(3) If the diffusive spread is measured at a number of positions extending far down-stream from the source a length l_1 analogous to the mean free path in kinetic theory of gases can be determined. It is anticipated that this will be some definite fraction of the mesh size M of the honeycomb or grid.

(4) Measurements of correlation between simultaneous values of the velocity at points distributed along a line can determine a length l_2 which measures the scale of turbulence from the standpoint of the Eulerian representation of fields of flow.

Both these lengths may be expected to be some definite fraction of the mesh length M, at any rate when the turbulence is not very small.

(5) A third length λ can be defined in relation to the dissipation of energy by the equation $\frac{\overline{W}}{\mu} = 15\frac{\overline{u^2}}{\lambda^2}$. This length may be taken to represent roughly the diameters of the smallest eddies into which the eddies defined by the scales l_1 or l_2 will break up.

(6) If the rate of dissipation is proportional to the cube of the velocity, as it is where the Reynolds's stresses are proportional to the squares of the turbulent components of velocity, λ is proportional to $\sqrt{\frac{l\nu}{u'}}$.

In turbulence due to a square mesh honeycomb of mesh length M, $\frac{\lambda}{M} = A\sqrt{\frac{\nu}{Mu'}}$, where A is a constant. This formula is inapplicable when Mu'/ν is small.

(7) Using this value for λ it is shown that the law of decay of turbulence is such that U/u' increases linearly with x in accordance with equation (59).

(8) λ is also directly connected with the correlation between simultaneous measurements of velocity at fixed points separated by a small distance. This correlation can be measured by suitable apparatus so that the theory can be verified experimentally.

(9) In isotropic turbulence the mean value of any quadratic expression of the space rates of change in the velocity is known when the mean value of any one of the terms in it which is not zero is known. This leads to the prediction, which might be verified experimentally, that if the correlation between a component of velocity at a fixed point O and that at a neighbouring variable point P is measured, the surfaces of equal correlation are prolate spheroids with P as centre, the long axis is $\sqrt{2}$ times the equatorial axis and is directed in the direction in which the velocity component is measured. This statement is identical in substance though not in form with that given on p. 439.

Concluding Remarks

Of these results and predictions (1), (2) and (3) are substantially identical with the conclusions put forward in 1921 in my paper, "Diffusion by Continuous Movements," where the suggestion that the diffusive power of turbulence should be used for the purpose of measuring the scale of turbulence and the turbulent components was first made. Recently experiments of this nature have been made by C. B. Schubauer* and (2) has been verified, as will be shown in Part IV of the present paper. Mr. Schubauer, however, worked quite independently of my previous work and indeed gives an empirical explanation of his experimental results. Conclusions (4) to (9) are, I believe, new. It will be shown in Part II that all these results, except (9), have now been verified experimentally and shown to be true. Experimental work is now in hand to test the truth of (9).

Statistical Theory of Turbulence—II

By G. I. Taylor, F.R.S.

(*Received July* 4, 1935)

Measurements of Correlation in the Eulerian Representation of Turbulent Flow

The methods described in Part I have been used by Mr. L. F. G. Simmons, of the National Physical Laboratory, to find experimentally the correlation between the turbulent components of velocity u_0 and u_y at two points distant y apart in a direction transverse to the stream. The measurements were made at mean speed U = 25 feet per second in a wind tunnel behind a honeycomb with 0·9-inch square mesh. The results are shown in fig. 1 where the ordinates are $R_y = \frac{\overline{u_0 u_y}}{\overline{u^2}}$ and the abscissae are the corresponding values of y. It will be seen that the R_y curve is apparently rounded at the top and that R_y falls to 0·08 at $y = 0 \cdot 38$ inches. No measurements were made beyond this point, but extrapolation seems to show that $R_y = 0$ when y is about 0·5 inches, *i.e.*, when y is slightly greater than $\frac{1}{2}$M.

* 'Rep. Nat. Adv. Ctee. Aero., Wash.,' No. 524 (1935).

Integrating the curve of fig. 1 the value of $l_2 = \int_0^{0\cdot 5''} R_y\,dy = 0\cdot 175$ inch, so that

$$l_2/M = 0\cdot 175/0\cdot 9 = 0\cdot 195. \qquad (1)$$

Another set of measurements of similar nature made in a parallel-wall tunnel 24·6 cm high by Prandtl and Reichardt is shown in fig. 1 of Part III, which is reproduced from their paper. Their method was not so suitable for measuring low values of R_y as that of Simmons, and their curves are not given for values of R_y less than 0·6.

The forms of the R_y curves near $y = 0$ will be discussed later.

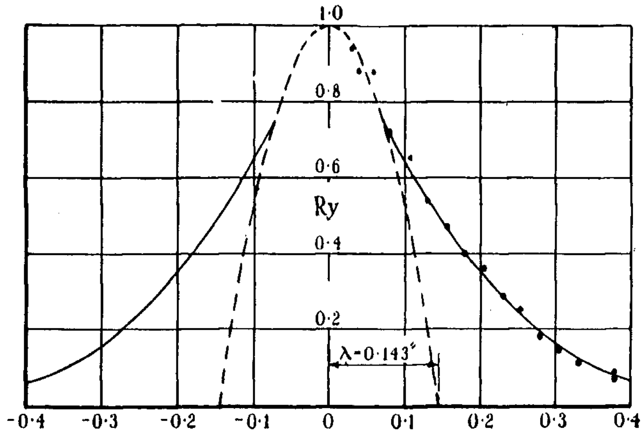


FIG. 1—Measured values of $R_y = \overline{u_0 u_y}/\overline{u_2}$ behind 0·9 inch by 0·9 inch honeycomb; $\overline{u^2} = 0\cdot 1015$ (ft sec)^2

DECAY OF TURBULENCE IN AN AIR STREAM BEHIND A GRID OR HONEYCOMB

Few observations of the decay of turbulence seem to have been made, and in most of them where observations from which the decay can be found have been made the experiment was concerned with other phenomena. In some such cases the necessary data for comparing with the theory developed in Part I of this paper were not all recorded.

SIMMONS AND SALTER'S MEASUREMENTS BEHIND GRIDS

The measurements contained in Table I were made by Simmons and Salter* by means of hot wires in the stream behind a grid consisting of

* 'Proc. Roy. Soc.,' A, vol. 145, p. 233 (*see* Table II, p. 233).

strips 0·5 inch wide forming square-mesh grid with 1-inch square holes so that M = 1·5 inches = 3·81 cm.

TABLE I—M = 1·5 INCHES

U ft sec	x inches	$\overline{u'^2}$ (ft sec)^2	u' ft sec	U/u'	Mu'/ν
5·19 7·2	6	1·23 2·02	1·27	4·88	1030
5·65 5·76	9	0·565 0·614	0·768	7·42	636
5·29	15	0·162	0·402	13·15	332

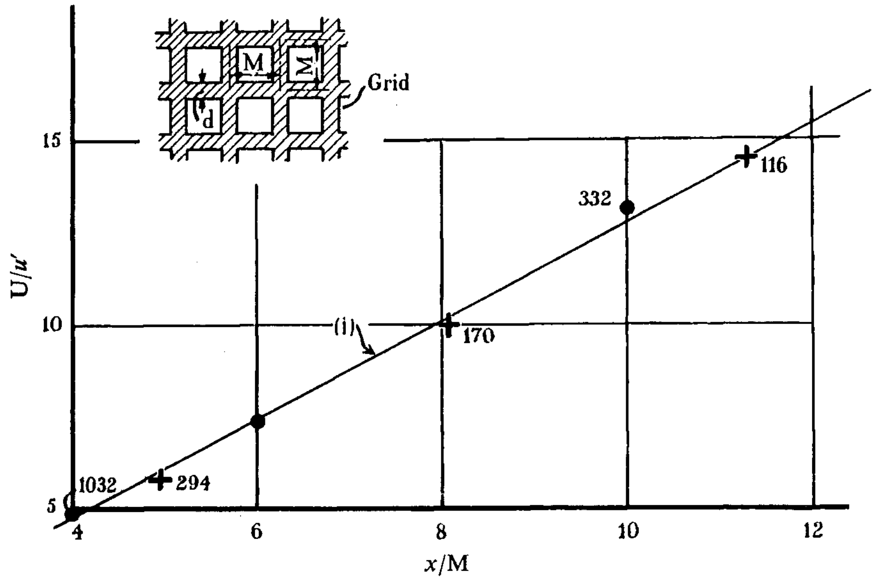


FIG. 2—Decay turbulence behind grid of flat strips; ● =1·5 inches; d =0·5 inches; + M = 0·62 inches; (i) U/u' = −0·7 + 1·32 x/M

In this table $u' = \sqrt{\overline{u^2}}$ and x is the distance behind the grid. At $x = 6$ and 9 inches the " wind shadow " had not disappeared, the two sets of readings being those obtained behind the middle of a strip and behind the middle of a hole respectively. The values of u' and U/u' in columns 4 and 5 are deduced from means of the two sets of figures in columns 1 and 3.

According to the theoretical prediction (*see* equation (59), Part I) U/u' should increase linearly with x. Fig. 2 shows U/u' plotted against x/M. It will be seen that the prediction is very nearly verified, the points lying very close to a straight line.

The value of the constant A which occurs in (59) can be calculated from any two of the three sets of measurements. Taking from Table I the measurements at $x = 6$ and 9 inches I find

$$A^2 = 3\cdot94 \text{ so that } A = 1\cdot98,$$

and taking the measurements at $x = 9$ and 15,

$$A^2 = 3\cdot49 \text{ so that } A = 1\cdot87.$$

The Reynolds number of the turbulence, namely, Mu'/ν, corresponding with the three stations are given in the last column of Table I.

It is of interest to compare the rate of decay of turbulence observed by Simmons and Salter with that which would result if the complete turbulence pattern remained constant through the range of observation from $x = 6$ inches to 15 inches, but the turbulent velocity decreased owing to viscosity. The turbulent velocity components in that case* would be proportional to e^{-ct}, so that

$$\frac{u'_{15}}{u'_6} = \left(\frac{u'_9}{u'_6}\right)^3. \tag{2}$$

The 3 appears as exponent in (2) because the time taken by the stream to travel the distance from $x = 6$ inches to 15 inches is three times as long as the time for $x = 6$ inches to 9 inches.

On the other hand, equation (59), Part I, can be reduced in this case to

$$\frac{u'_{15}}{u'_6} = \frac{u'_9/u'_6}{3 - 2u'_9/u'_6}. \tag{3}$$

From Table I it will be seen that $u'_9/u'_6 = 0\cdot605$, thus the supposition that turbulent motion decays by the direct action of viscosity without any increase in the diameter of the smallest eddies leads to the prediction that

$$u'_{15}/u'_6 = (0\cdot605)^3 = 0\cdot22. \tag{A}$$

The present theory leads to the prediction that

$$u'_{15}/u'_6 = \frac{0\cdot605}{3\cdot2\,(0\cdot605)} = 0\cdot338. \tag{B}$$

The observed value is

$$u'_{15}/u'_6 = \frac{0\cdot402}{1\cdot27} = 0\cdot317. \tag{C}$$

Comparing these three values ((A), (B), (C)) it will be seen that Simmons and Salter's observations are in fairly good agreement with the

* Taylor, 'Phil. Mag.,' vol. 46, p. 671 (1923).

present theory but do not agree with theories of dissipation which do not allow for increasing size of the smallest eddies with decrease in Mu'/ν.

Some further unpublished observations made by Mr. Simmons with a grid geometrically similar to his $1\frac{1}{2}$-inch grid, but of smaller mesh (M = 0·62), are also shown in fig. 2. It will be seen that when plotted non-dimensionally (*i.e.*, using x/M as abscissae) the points lie very close to those which represent the experiments with the larger grid. Thus the prediction that the constant A of equation (59), Part I, is a constant for all similar grids is verified. The question whether A is a universal constant, as is predicted in Part I, for all square-mesh grids irrespective of the ratio of the width of the bars of the grid to the mesh remains open so far as these observations are concerned.

Dryden's Observations Behind Honeycombs

The first reliable observations on the decay of turbulence behind a honeycomb are contained in a report by Dryden.* In that report the size of the honeycomb was not given, but Dr. Dryden has very kindly supplied me with the necessary data. The honeycomb was hexagonal, the distances between opposite faces being 3 inches. Taking M = 3·0 inches the observed values are given in Table II.

TABLE II

x/M	21	38	58
U/u'	43·5	62·5	83·3

If U = 20 ft/sec Mu'/ν lies between 1000 and 2000. Similar observations behind a honeycomb of 3-inch circular tubes arranged in hexagonal piling gave the results shown in Table III.

TABLE III

x/M	21	38	58
U/u'	62·5	77	100

These results being obtained with honeycombs of circular and hexagonal tubes are not strictly comparable with those obtained with honeycombs or grids arranged in a pattern of squares. The difference, however, may be expected to be slight, and they are included because, as will be seen in fig. 6 where they are shown in graphic form, they illustrate how accurately the theoretical linear law of increase in U/u' is borne out in practice.

* 'Rep. Nat. Adv. Cttee. Aero. Wash.,' No. 342, *see* also Report No. 392.

N.P.L. Observations Behind Grids and Honeycombs

Recently Simmons and Salter have made some more measurements of U/u', using a hot-wire compensating circuits and a thermo-junction milliammeter to measure $\overline{u^2}$ in the manner explained in Part I. A set of observations made behind a grid of round bars 0·875 inch diameter arranged in a 3-inch square mesh* is shown in fig. 3. In the work previously described the average value of U/u' taken over a long series of observations made at the same section of the tunnel were given. In

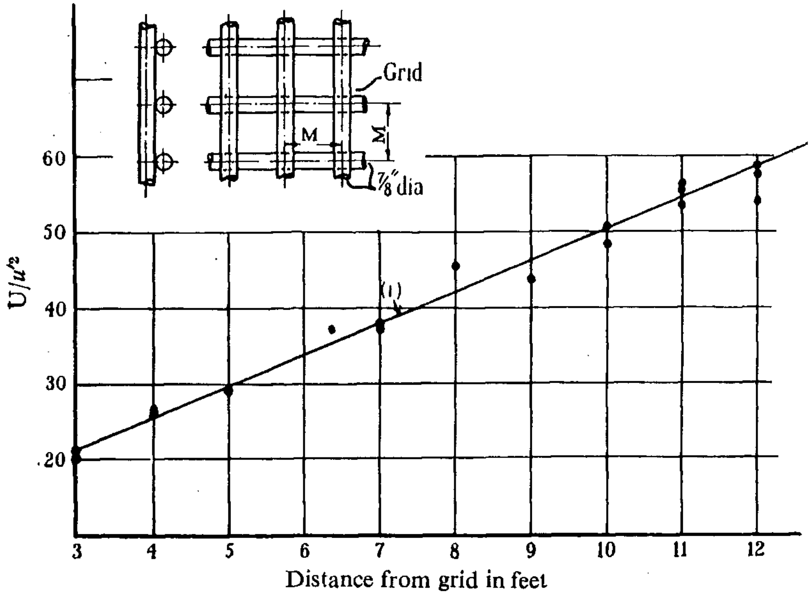


FIG. 3—Decay of turbulence behind a grid of round bars. $U = 19{\cdot}5$ ft/sec; $M = 3$ inches; $A = 2{\cdot}20$; (i) $U/u' = 8{\cdot}9 + 1{\cdot}035\, x/M$

fig. 3 the actual observations or means of a small number of observations are shown. It will be seen that the linear law of increase in U/u' is in good agreement with the experiments. The line shown in the figure is $\frac{U}{u'} = 8{\cdot}9 + 1{\cdot}035\frac{x}{M}$, which agrees with equation (59) of Part I if $A = 2{\cdot}20$.

A similar set of measurements in a 4-foot wind tunnel behind a 3-inch square-mesh honeycomb is shown in fig. 4. Again it will be seen that the measurements confirm the theoretical linear law of increase in U/u',

* As is indicated in fig. 3, this grid consisted of two sets of parallel rods in contact so that the central plane of one set is a distance d further down-stream than the other.

the line in this case being $\frac{U}{u'} = 12 \cdot 6 + 1 \cdot 230 \frac{x}{M}$, which agrees with (59), Part I, if $A = 2 \cdot 05$.

Observations behind a 0·9-inch by 0·9-inch honeycomb in a 1-foot tunnel at 5 ft per sec gave a well-defined straight line

$$\frac{U}{u'} = 5 \cdot 5 + 1 \cdot 30 \frac{x}{M}. \tag{4}$$

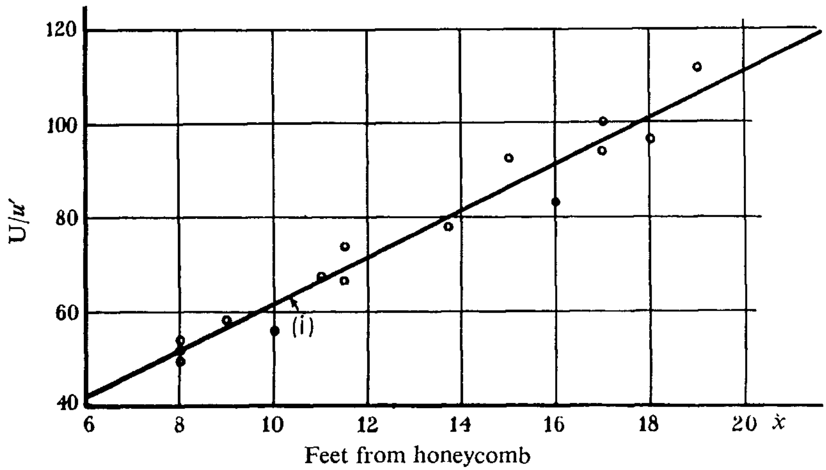


FIG. 4—Decay of turbulence behind a square-mesh honeycomb. $U = 20 \cdot 2$ ft/sec; $M = 3$ inches; $A = 2 \cdot 05$; (i) $U/u' = 12 \cdot 6 + 1 \cdot 23\, x/M$

DRYDEN'S OBSERVATIONS BEHIND GRIDS

Recently Dr. H. L. Dryden has made a series of observations of u'/U at various distances behind a set of nearly similar square grids consisting of round bars each about 0·193 of the mesh length. The values of M were M = 5 inches, $3\frac{1}{4}$ inches, 1 inch, $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, $\frac{1}{8}$ inch. Dr. Dryden kindly set forth his results for me in the diagram shown in fig. 5 where the abscissae are x/d, d being the diameter of the bars of the grid and x the distance down-stream from the grid. The numbers attached to the points are the mesh length M in inches. The straight line drawn in the diagram which passes nearly through the points corresponding with the M = 5 inches and $3\frac{1}{4}$ inches is

$$\frac{U}{u'} = 7 + 0 \cdot 25 \frac{x}{d},$$

and since $d = 0 \cdot 193$ M this is equivalent to

$$\frac{U}{u'} = 7 + 1 \cdot 30 \frac{x}{M}. \tag{5}$$

This line fairly represents the observations up to $U/u' = 100$, but for higher values of U/u' the turbulence does not decrease so rapidly as the theoretical formula would lead one to expect.

The points on the extreme right of the diagram, fig. 5, correspond with very low values of Mu'/ν, namely, 7 and 14. This is well below the lowest value of Mu'/ν at which the linear law would be expected to hold, but, as has been pointed out, the rate of decay for a given scale of turbulence at low values of Mu'/ν must be *greater* than that indicated by the linear law. Fig. 5 shows that in Dr. Dryden's tunnel the turbulence dies away more slowly than the linear law when the turbulence is small. This indicates

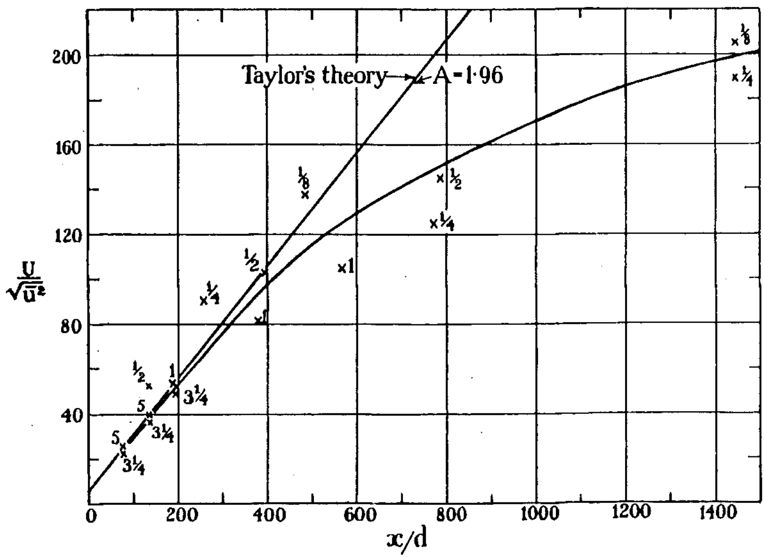


FIG. 5—Decay of turbulence behind a grid of round bars (H. L. Dryden)

that the large-scale turbulence which is in the air stream before striking the grid can pass through it. Near the grid this is masked by the more violent disturbances produced by the grid itself. As the distance downstream from the grid increases the large-scale turbulence which has passed through it decreases more slowly than that produced by the grid and eventually completely obliterates the effect of the grid.

It seems likely that a honeycomb with long cells might be more likely to reduce the pre-existing turbulence to a definite scale than a grid of round bars, and in fact the theoretical linear law of decay seems to be obeyed very consistently in the disturbances behind honeycombs.

Summary of Results on Decay of Turbulence

All these results are collected together in fig. 6 and the equations to the straight lines which pass most nearly through the observed points are collected together in Table IV. Dr. Dryden's results with small-mesh honeycombs, M = 1 inch, ½ inch, ¼ inch, ⅛ inch, have not been included because the corresponding values of LU/ν are so small that the theory

TABLE IV

Authority	—	M inches	Equation to line in fig. 6	A
Dryden	Honeycomb of round 3-inch tubes	3·0	$U/u' = 37{\cdot}1 + 1{\cdot}056\, x/M$	2·18
	Hexagonal honeycomb	3·0	$U/u' = 21{\cdot}1 + 1{\cdot}078\, x/M$	2·16
	5-inch square-mesh grid of round bars $d/M = 0{\cdot}2$	5·0	$U/u' = 9{\cdot}5 + 1{\cdot}110\, x/M$	2·12
	3¼-inch square-mesh grid of round bars $d/M = 0{\cdot}2$	3·25	$U/u' = 8{\cdot}8 + 1{\cdot}072\, x/M$	2·16
Simmons and Salter	1½-inch square-mesh grid of flat slats $d/M = 0{\cdot}33$	1·50	$U/u' = -0{\cdot}7 + 1{\cdot}32\, x/M$	1·95
	0·62-inch grid of slats	0·62		
	Square-mesh honeycomb	0·90	$U/u' = 5{\cdot}5 + 1{\cdot}30\, x/M$	1·96
	Square-mesh honeycomb	3	$U/u' = 12{\cdot}6 + 1{\cdot}23\, x/M$	2·05
	Square-mesh grid of round bars $d/M = 0{\cdot}29$	3	$U/u' = 8{\cdot}9 + 1{\cdot}035\, x/M$	2·20

could not be expected to apply to them, and also because, as has been pointed out, there is evidence that the disturbances already in the stream pass through the grids and, when M is small, become more important than those produced by the grid.

The values of A calculated by using the relationship

$$\frac{U}{u'} = \text{constant} + \frac{5}{A^2}\frac{x}{M}$$

as identical with the equations in column 4 of Table IV are given in column 5. It will be seen from fig. 6 that though U/u' ranges from 4·9 to 110, and M from 0·62 inch to 5 inches all the observed values of A given in column 5 of Table IV lie in the range from 1·95 to 2·20. This appears to confirm the accuracy of the theoretical prediction made in Part I that A is a universal constant for turbulence produced or controlled by any type of square grid or honeycomb.

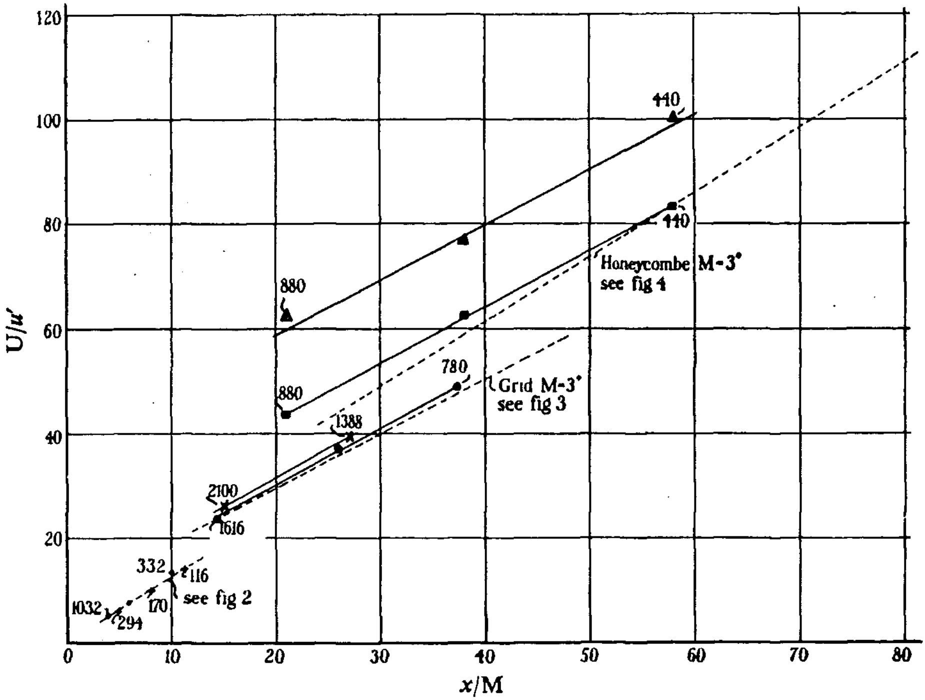


FIG. 6—Collected results of turbulence behind grids and honeycombs.

Dryden:
- × 5-inch grid, M = 5 inches, d = 1 inch.
- ⊙ $3\frac{1}{4}$-inch grid, M = 3·25 inches, d = 0·65 inch.
- ⊡ Hexagonal honeycomb, M = 3 inches.
- Δ Honeycomb of 3-inch tubes, M = 3 inches.

Simmons and Salter:
- + grid, M = 0·62 inch in 4-inch pipe.
- • grid, M = 1·5 inches in 1-foot tunnel.

EXPERIMENTAL VERIFICATION OF CONNECTION BETWEEN DISSIPATION OF ENERGY AND INITIAL CURVATURE OF R_y CURVE

In Part I it is predicted that near the origin the correlation curve, R_y, will coincide with the parabola

$$y = \lambda\sqrt{1 - R_y}, \qquad (6)$$

where $\lambda^2 = W/(15\ \mu\overline{u^2})$ and the dissipation is W. It has now been shown that the dissipation behind a honeycomb or grid is in good agreement with observation provided

$$\frac{\lambda}{M} = A\sqrt{\frac{\nu}{u'M}}, \tag{7}$$

where A is a constant whose value lies between 1·95 and 2·2. The correlation observations shown in fig. 1 were made in the air stream behind a square-mesh honeycomb for which M = 0·9 inch. It is possible therefore to test the theoretical prediction in this case because the value of u'^2 or $\overline{u^2}$ was measured at the same time as the correlation coefficient R_y. In ft/sec units $\overline{u^2}$ was 0·1015 so that $u' = 0{\cdot}3185$ ft per sec = 9·70 cm/sec.

Taking A as 2·0, (7) gives

$$\frac{\lambda}{M} = 2{\cdot}0\sqrt{\frac{0{\cdot}14}{(9{\cdot}70)\,(0{\cdot}9 \times 2{\cdot}54)}} = 0{\cdot}159. \tag{8}$$

Hence $\lambda = 0{\cdot}9\,(0{\cdot}159) = 0{\cdot}143$ inches. The parabola $y = 0{\cdot}143\sqrt{1 - R_y}$ is shown as a broken line in fig. 1. It will be seen that it does in fact pass through the observed points near the summit of the curve. The theoretical prediction is therefore verified in this case.

In conclusion I wish to express my thanks to Mr. L. F. G. Simmons for permission to publish figs. 1, 3, and 4, and to Dr. H. L. Dryden for supplying fig. 5.

Statistical Theory of Turbulence
III—Distribution of Dissipation of Energy in a Pipe over its Cross-Section

By G. I. TAYLOR, F.R.S.

(*Received July* 4, 1935)

The success of the theory of turbulence given in Part I, in predicting the connection between the rate of dissipation of energy in turbulent motion and the shape of the R_y correlation curve, suggests that it may be possible to use measurements of correlation in order to find out how the dissipation of energy is distributed over a field of turbulent flow. This is a fundamental question in any theory of turbulent motion which attempts to penetrate beyond the stage of empirical assumptions.

By a fortunate coincidence the necessary observations already exist for the analysis of one case of flow from this point of view. The rate of dissipation of energy of turbulent flow is

$$W = 7 \cdot 5\ \mu \overline{\left(\frac{\partial u}{\partial y}\right)^2} = 15\ \mu \frac{\overline{u^2}}{\lambda^2}. \qquad (1)$$

Reference was made in Parts I and II to the correlation measurements of Prandtl and Reichardt in air flowing under pressure between two parallel planes 24·6 cm apart. In these experiments measurements were taken with one fixed hot wire at seven positions, namely 1, 2, 3·5, 5·5, 8, 10·5, and 12·3 cm from the wall. Corresponding with each position of the fixed wire a traverse of the second hot wire was made across the channel and the variable currents produced by the two wires caused a spot of light to oscillate over a plate, one of the wires producing a horizontal movement and the other a vertical movement. The darkened areas thus produced were roughly elliptical. With very high correlation between the velocities at the two points these ellipses became very narrow lines at 45° to either axis. The degree of correlation was found by measuring the ratio of the axes of the ellipses. The correlation coefficients measured in this way are those represented in Part I by the symbol R_y. They rise to 1·0 when the movable wire is in contact with the fixed wire and fall away from 1·0 as the two wires separate. This is shown in fig. 1, which is a reproduction from Prandtl and Reichardt's paper of the curves which represent their results.

In each case four observations were made very close to the fixed wire so that the forms of the correlation near their summits were well determined.* It will be seen that these summits are rounded in the manner predicted in Part I.

By measuring on an enlarged reproduction of fig. 1 I find the values for $(1 - R_y)$ and y at various distances from the wall which are given in Table I.

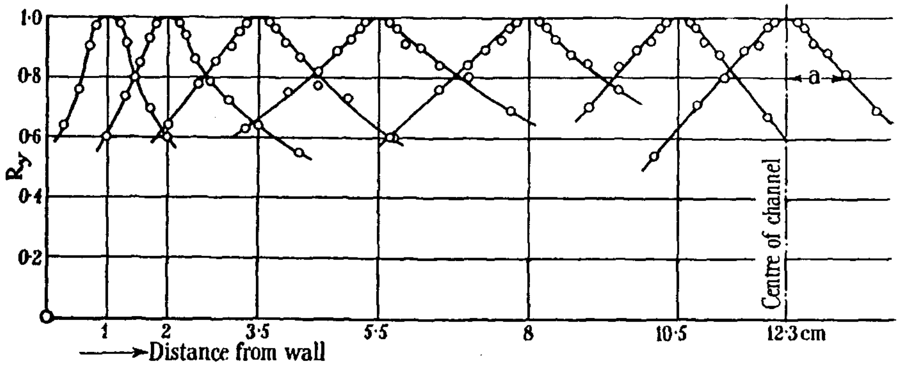


FIG. 1—Prandtl and Reichardt's correction measurements in a 24·6-cm channel

TABLE I

Distance of fixed hot wire from wall cm	12·3 (Y = 0)		10·5 (Y = 1·8)		8·0 (Y = 4·3)	
y mm	3·04	1·93	3·00	2·01	3·10	1·84
$1 - R_y$	0·0316	0·0132	0·0317	0·0132	0·0346	0·0138
λ^2 (cm²)	2·98	2·82	2·82	3·06	2·75	2·48
Mean λ^2 ((cm²)	2·90		2·94		2·62	
Distance from wall cm	5·5 (Y = 6·8)		3·5 (Y = 8·8)		2·0 (Y = 10·3)	
y mm	3·00	1·99	2·84	1·93	3·00	2·01
$1 - R_y$	0·0349	0·0132	0·0407	0·0156	0·0627	0·0189
λ^2 (cm²)	2·59	3·03	1·96	2·36	1·44	2·12
Mean λ^2	2·81		2·16		1·78	
Distance from wall cm	1·0 (Y = 11·3)					
y mm	3·06	2·034				
$1 - R_y$	0·090	0·0234				
λ^2 (cm²)	1·06	1·77				
Mean λ^2 (cm²)	1·42					

* Prandtl's method is specially suitable for determining the small deviation of the correlation from 1·0.

In Table I the figures given for y are the means of the distances of the nearer and of the farther pairs in the groups of four points close to each of the positions of the fixed hot wires (*see* fig. 1). The values of λ^2 are calculated from the formula

$$\lambda^2 = y^2/(1 - R_y), \tag{2}$$

and at the bottom of the table the mean value of λ^2 is given for each position.

Calculation of Dissipation from Correlation and Turbulence Measurement

The values of $(u'/U_{max})^2$ have been measured by Wattendorf and Kuethe* for a parallel-walled channel 5 cm deep when $U_{max} = 9 \cdot 6$ metres per sec.

Their results are given in curve ii, fig. 2. It will be seen that at the centre of the channel $(u'/U_{max})^2 = 0 \cdot 00135$ and that its value increases towards the walls. The Reynolds number of these measurements was greater than that of the channel in Prandtl's measurements in the ratio

$$\frac{960}{114} \times \frac{5}{24 \cdot 6} = 1 \cdot 67 : 1,$$

but Wattendorf and Kuethe also measured the change in u'/U_{max} at the centre of the pipe with change in U_{max}. Their results do not extend to quite so low a Reynolds number as Prandtl's, but a very small extrapolation of their curve suggests that at the Reynolds number of Prandtl's experiment u'/U_{max} would be about $0 \cdot 039$, so that $(u'/U_{max})^2 = 0 \cdot 00152$.

Wattendorf and Kuethe give reasons for believing that the values of u'/U_{max} should be proportional to† v_x/U_{max} as U_{max} varies, and their observations of turbulence, taken in the middle of their channel, confirm this view.

If this law of variation in u'/U_{max} with Reynolds's number is correct it must be equally true for all positions in the cross-section of the channel. Accordingly to estimate the values of u'/U_{max} at the Reynolds number of Prandtl's correlation measurements it is necessary to multiply Wattendorf and Kuethe's values obtained at $9 \cdot 6$ metres per sec in a 5-cm channel by the factor $\frac{0 \cdot 00152}{0 \cdot 00135} = 1 \cdot 125$. In this way the curve B, fig. 2, is obtained from Wattendorf and Kuethe's curve A.

* 'Pasadena Publication,' No. 45, Guggenheim Aeronautics Laboratory.
† v_x is defined as $\sqrt{\tau_0/\rho}$ where τ_0 is the surface stress at the wall.

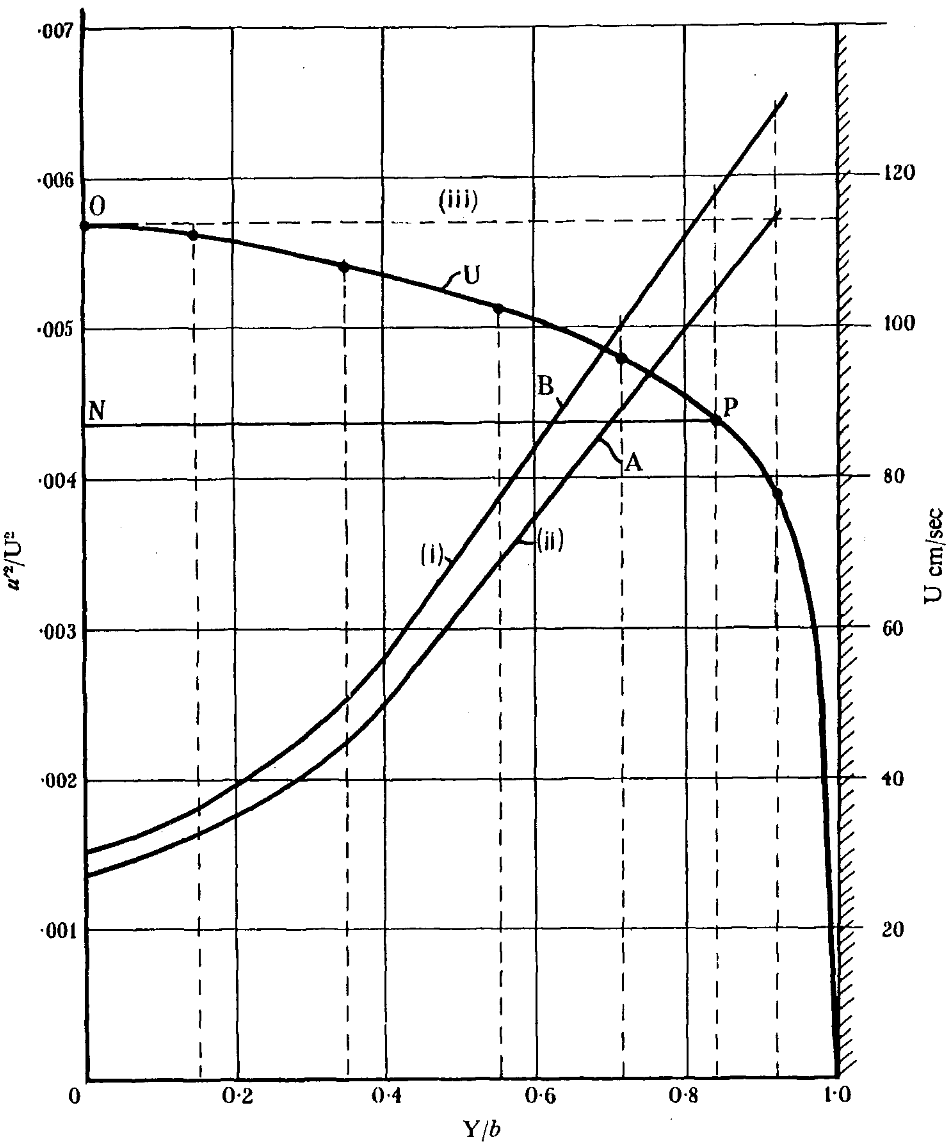


FIG. 2—Velocity and u components of turbulence in a 24·6-cm channel. (i) u'^2/U^2 for 24·6-cm channel when $U_{max} = 114$ cm/sec; (ii) u'^2/U^2, Wattendorf and Kuethe's measurements for 5-cm channel when $U^{max} = 960$ cm/sec; (iii) U_{max} 114 cm/sec

From curve B the values of u'^2/U^2_{max} appropriate to the positions where Prandtl made correlation measurements were taken. These are given in column 2 of Table II. Multiplying by $(114)^2$ gives u'^2 in $(cm/sec)^2$ (column 3). The values of λ^2 taken from Table I are given in column 4. The figures in column 1 are the distances Y from the centre of the channel.

Taking μ as $1 \cdot 80 \times 10^{-4}$ c.g.s., $15\ \mu = 2 \cdot 7 \times 10^{-3}$. The values of $15\mu u'^2/\lambda^2$ are given in column 5. This is the rate of dissipation of turbulent energy into heat per cc per sec.

The total dissipation of energy in a column 1 sq. cm section between the central plane and the parallel plane distant Y from the centre is $15\ \mu \int_0^Y \frac{u'^2}{\lambda^2} dY$. The values of this are given in column 7, Table II.

Calculation of Rate of Production of Energy of Turbulent Motion from Surface Friction and Velocity Profile

The dissipation formula (1) gives the rate of dissipation per unit volume of kinetic energy of turbulent motion into heat. It is possible to deduce the total dissipation of energy in a pipe or channel if the surface friction and the mean velocity of the air is known, but it is not possible to deduce from such measurements or from measurements of the distribution of velocity across the section how the dissipation of energy is distributed across the section.

On the other hand, the rate of transformation of energy of mean flow into energy or turbulent flow can be calculated in this way.

Fig. 2 shows the observed distribution of velocity in the channel when the correlation observations shown in fig. 1 were made. The total dissipation of energy per cm length of the channel in a section 1 cm wide is $2\tau_0 U_m$ where U_m is the mean velocity in the channel and τ_0 is the stress on the walls. The tangential stress τ at distance Y from the centre is $\tau_0 Y/b$, where $2b$ is the width of the channel, namely, $24 \cdot 6$ cm. The rate at which the fluid between the middle of the channel and the plane distant Y from the centre does work on the fluid contained between this plane and the walls is $U\tau_0 Y/b$. The rate at which the pressure gradient does work on the fluid is $\frac{\tau_0}{b}\int_0^Y U\,dY$, so that the difference between these two, namely,

$$\frac{\tau_0}{b}\left[\int_0^Y U\,dY - UY\right] \text{ or } -\frac{\tau_0}{b}\left[\int_0^Y Y\frac{dU}{dY}\,dY\right], \tag{3}$$

must be equal to the (rate of dissipation of energy by viscosity acting on the mean flow) + (rate of transformation of energy of mean flow into

TABLE II

1	2	3	4	5	6	7	8
Y cm	$\frac{u'^2}{U^2_{max}}$	u'^2	λ^2	$15\,\mu \frac{u'^2}{\lambda^2}$ ergs per sec per cc	$\frac{\tau_0}{b} Y \frac{dU}{dY}$	Total dissipation from 0 to Y $15\,\mu \int_0^Y \frac{u'^2}{\lambda^2}\,dY$ ergs per sec	$\frac{\tau_0}{b} \int_0^Y Y \frac{dU}{dY}\,dY$ ergs per sec
0	0·00152	19·8	2·90	0·0184	0		
1·8	0·00182	23·6	2·94	0·0216	0·0059	0·105	0·038
4·3	0·00252	32·8	2·62	0·0337	0·0230	0·208	0·124
6·8	0·00386	50·2	2·81	0·0483	0·0572	0·337	0·302
8·8	0·00501	65·1	2·16	(0·0807)	0·112	0·485	0·512
10·3	0·00589	76·5	1·78	(0·1160)	0·198	0·621	0·812
11·3	0·00644	83·8	1·42	(0·1590)	0·470		

energy of turbulent flow). The former of these is $\int_0^Y \mu \left(\frac{dU}{dY}\right)^2 dY$, so that the total rate of transformation of energy of mean flow into energy of turbulent flow between the centre and the plane is

$$-\frac{\tau_0}{b}\int_0^Y Y\frac{dU}{dY}dY - \int_0^Y \mu\left(\frac{dU}{dY}\right)^2 dY. \qquad (4)$$

The rate of transformation of energy of mean flow into energy of turbulent flow per unit volume is found by differentiating (4); it is

$$-\frac{\tau_0}{b}Y\frac{dU}{dY} - \mu\left(\frac{dU}{dY}\right)^2. \qquad (5)$$

In all cases of turbulent flow $\mu\left(\frac{dU}{dY}\right)^2$ is small compared with $\frac{\tau_0}{b}Y\frac{dU}{dY}$ except in a thin layer near the wall. At the wall itself they are equal to one another (but of opposite sign) if the wall is smooth, so that the expression (4) tends to zero at the wall. Except in a thin layer close to the wall, therefore, the rate of transformation per unit volume of mean energy into turbulent energy may be taken as

$$-\frac{\tau_0}{b}Y\frac{dU}{dY}. \qquad (6)$$

In this expression τ_0 can be obtained from the Kármán-Prandtl friction formula

$$\frac{U}{v_x} = 5\cdot5 + 5\cdot75\log_{10}\left[\frac{(b-Y)\,v_x}{\nu}\right], \qquad (7)$$

where $\tau_0 = \rho v_x^2$.

Taking from the curve of fig. 2 the observed value $U = 95\cdot6$ cm per sec at $3\cdot5$ cm from the wall, I find from the formula (7) that if $\nu = 0\cdot14$, $v_x = 5\cdot39$ cm per sec. Taking $\rho = 0\cdot00123$, gives $\tau = 0\cdot0357$ dynes, so that $\tau_0/b = 0\cdot00290$. dU/dY can be found by taking the slope of the tangents to the velocity profile in fig. 2 and the values of $\frac{\tau_0}{b}Y\frac{dU}{dY}$ calculated in this way are given in column 6 of Table II.

Comparing column 5, which gives with the rate of dissipation per cc, with column 6, which gives the rate of degradation of mean energy into turbulent energy, it will be seen that from the centre of the channel to about $Y = 6$ cm the rate of dissipation is greater than the rate of transformation from energy of mean motion to energy of turbulent motion. From $Y = 6$ to the wall the rate of transformation of mean energy into turbulent energy is greater than the rate of dissipation. This result is shown graphically in fig. 3.

LIMITATION TO APPLICATION OF DISSIPATION FORMULA

The dissipation formula (1) is only accurate when the turbulence is isotropic. It has been shown by Fage that the turbulence in the central

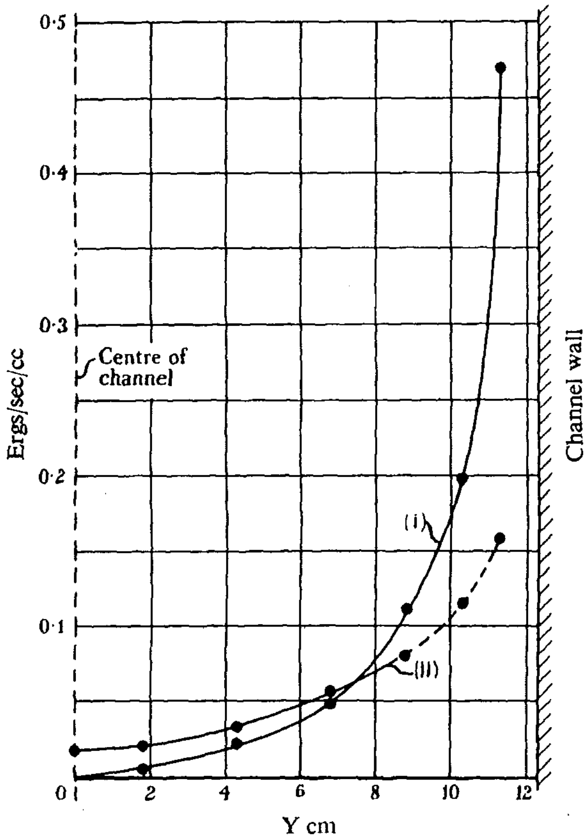


FIG. 3—(i) $-\frac{\tau_0}{b} Y \frac{du_1}{dy}$; (ii) $15\mu \frac{\overline{u^2}}{\lambda^2}$

region of a square pipe is approximately isotropic, at any rate so far as the equality of the three components of turbulent motion are concerned. On the other hand, near the wall of the pipe the transverse component of turbulent velocity becomes greater than the longitudinal component or the component normal to the surface. Roughly it may be taken that the region of large divergence from the isotropic condition stretches from about $r/a = 0 \cdot 7$ to the wall. If the results for a square pipe can be taken as representative of what would be found with a two-dimensional channel, it may be anticipated therefore that the dissipation formula (1) will be seriously in error when Y/d is less than $0 \cdot 7$. In Prandtl's channel $d = 12 \cdot 3$ cm, so that serious errors in the dissipation formula may be expected when $Y > 8 \cdot 6$ cm. For this reason the values given in column 5 of Table II for $Y = 8 \cdot 8$, $10 \cdot 3$, and $11 \cdot 3$ must be treated with reserve. They are enclosed in brackets to distinguish them from the more reliable figures, for $Y = 0$ to $Y = 6 \cdot 8$, and the corresponding part of the curve of fig. 3 is shown as a broken line.

Total Dissipation in Central Region of Channel

If formula (1) could be used accurately up to the wall the total dissipation must be equal to the total work done so that

$$-\frac{\tau_0}{b}\int_0^b Y\frac{dU}{dY}\,dY = \int_0^b \mu\left(\frac{dU}{dY}\right)^2 dY + 15\mu\int_0^b \frac{u'^2}{\lambda^2}dY.$$

It has been pointed out that it is only in the laminar layer near the wall that $\mu\left(\frac{dU}{dY}\right)^2$ is appreciable, and in that region the change in $-\frac{\tau_0}{b}\int_0^Y Y\frac{dU}{dY}\,dY$ is likely to be about equal to the change in $\int^Y \mu\left(\frac{dU}{dY}\right)^2 dY$, consequently it seems that we should expect the values of $-\frac{\tau_0}{b}\int_0^Y Y\frac{dU}{dY}\,dY$ and $15\,\mu\int_0^Y \frac{\overline{u^2}}{\lambda^2}dY$ to approximate to one another near the wall. The values of these two integrals are given in columns 7 and 8 of Table II and are shown graphically in fig. 4. It will be seen that over the whole region in which the dissipation formula may be expected to be valid the dissipation is greater than the rate of transformation from mean energy to turbulent energy, but that at the point $Y = 8 \cdot 6$ cm where the dissipation formula begins to be seriously in error the two curves are approaching one another. The fact that they cross one another at $Y = 9 \cdot 6$ cm may be due to the error in the dissipation formula when $Y > 8 \cdot 6$, or it may be due to the fact that the dissipation in the layer very close to the wall is necessarily greater than that in a laminar layer in which the velocities are parallel to the length of the channel. Fage has shown, in fact, that ultra-microscopic particles very close to a smooth surface past which fluid is moving in turbulent motion have transverse components as great as their total longitudinal components, so that in the layer immediately adjacent to a smooth surface the rate of dissipation must be *greater* than $\tau_0 \dfrac{dU}{dY}$.

The fact that the two curves of fig. 4 are so close to one another at the point where the dissipation formula ceases to apply certainly affords experimental confirmation of the accuracy of the conceptions on which

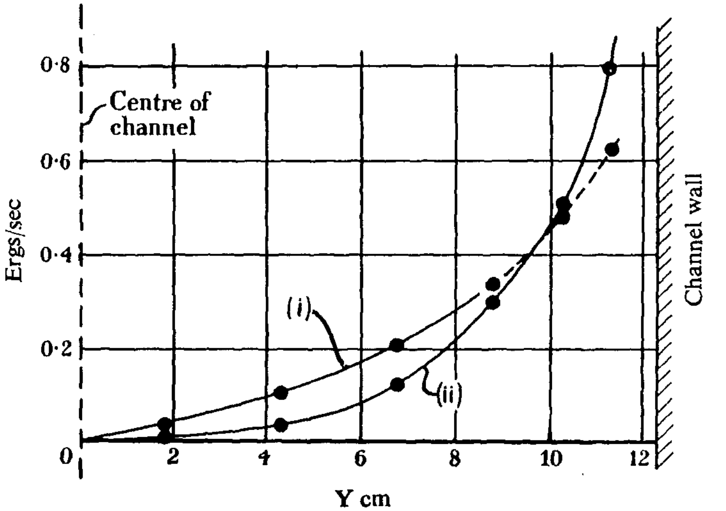


FIG. 4—(i) $15\,\mu \int_0^y \frac{\overline{u^2}}{\lambda^2}\,dy$ (ii) $-\frac{\tau_0}{b}\int_0^y y\,\frac{dU}{dy}\,dy$.

the theory is based, for it must be remembered that all the physical measurements which were used in calculating the dissipation curve are of a totally different character from that used in calculating the rate of transformation of the work done on the mean flow into energy of turbulent flow. In particular it seems that the numerical factor 15 which occurs in (1) has some experimental as well as theoretical justification.

Statistical Theory of Turbulence
IV—Diffusion in a Turbulent Air Stream

By G. I. TAYLOR, F.R.S.

(*Received July* 4, 1935)

It was pointed out in Part I that experiments on the spread of heat from a line source (*e.g.*, an electrically heated wire) in a turbulent air stream may be expected to give two elements of the statistical specification of turbulence. If the spread is measured near the source the value of the mean transverse component of velocity $\sqrt{\overline{v^2}}$, or v' in the notation of Part I, can be found. If the spread is examined further down-stream it should be possible to analyse the results to find the correlation function R_ξ, which is the principal element of the representation of turbulence in the Lagrangian system.

SPREAD OF HEAT NEARER LINE SOURCE

Recently the spread of heat from a heated wire in a wind tunnel has been measured at points near to the source by Schubauer.* The stream was made turbulent by means of grids of round bars arranged in square pattern. Their diameters were 1/5 of the mesh length and M varied from 5 inches to $\frac{1}{4}$ inch. The width of the heat wake was found by measuring the angle subtended at the source by the two positions where the temperature rise was half that in the centre of the wake. This angle, denoted by α, depends partly on the amount of turbulence and to a less extent on the spread of heat due to the thermal conductivity of the air. By assuming that the effect of turbulence is to communicate to the air an eddy conductivity β, which is additive to, and obeys the same law as, true thermal conductivity, a virtual angle α_{turb} can be deduced by the relation

$$\alpha^2_{turb} = \alpha^2 - \alpha^2_0, \tag{1}$$

where α_0 is the value which α would have if true thermal conductivity were the only agency involved in diffusing the heat. The angle α_{turb} calculated in this way is taken to be a measure of turbulence. It was discovered experimentally that the value of α_{turb} so found is independent of the distance between the hot wire and the station at which the spread

* 'Rep. Nat. Adv. Ctee. Aero., Wash.,' No. 524 (1935).

of the heat was measured, provided that distance is small. The value of the eddy conductivity β was calculated from the observations and found to be proportional to the distance x between the hot wire and the point at which the spread of the heat wake was measured. This relationship is a necessary result of the formal application of ordinary expression for the width of a wake in a conducting air stream to a case when the width of the heat wake is proportional to x; it is clear, however, that eddy conductivity β in this case can have no physical meaning because in obtaining his expression* for the width of a wake, namely, $\alpha^0 = 190 \cdot 8 \sqrt{\frac{\kappa + \beta}{\rho c U x}}$, Schubauer implicitly assumes that the conductivity is constant over length x, but depends on the distance of the station where measurements were made from the source of heat. Actually, if β is to have any meaning it must depend only on the state of the turbulent air, not on the position of the measuring apparatus.

Though Schubauer's derivation of equation (1) seems to be open to criticism on the ground that it is based on the hypothesis that the effect of turbulence can be represented by a virtual eddy conductivity β, which has been shown to have no physical meaning, yet it seems that equation (1) can be justified quite independently of any such conception, if the view of turbulent diffusion put forward in Part I is accepted. According to this view, the effect of turbulence near the source is to cause a linear rate of increase in $\sqrt{\overline{Y^2}}$ with x in accordance with the equation

$$\frac{\sqrt{\overline{Y^2}}}{x} = \frac{v'}{U}, \tag{2}$$

where $v' = \sqrt{\overline{v^2}}$.

It is not possible from (2) alone to calculate Dr. Schubauer's α_{turb}, but Simmons and Salter† and also Townend‡ have shown that the turbulent components of velocity are distributed according to the error law. Equation (2) is applicable during the period when R_ξ is nearly equal to $1 \cdot 0$. If $R_\xi = 1$, $Y = xv/U$, so that the frequency distribution of Y for the heated particles near the source is identical with the frequency distribution of v, *i.e.*, it obeys the error law.

If all particles emerging from the hot source have the same initial high temperature, the distribution of temperature in the wake must be the same as the frequency distribution of Y. Thus the temperature distribution

* *Loc. cit.*, p. 3, equation (4).

† 'Proc. Roy. Soc.,' A, vol. 145, p. 212 (1934).

‡ 'Proc. Roy. Soc.,' A, vol. 145, p. 180 (1934).

across the wake must be the normal error curve which also happens to be that which would be produced by true conductivity. Dr. Schubauer, in fact, finds that the temperature in the wake near the source is distributed according to the normal error law.

In the region where R_ξ is nearly equal to 1·0 we may regard the actual distribution of temperature as being produced (*a*) by the movements of the heated particles to distance $Y = \frac{U}{v}x$ from the source, Y being distributed according to the error law, and (*b*) diffusion by true conductivity from each of the heated particles. Since the distribution of temperature in each of these heated areas is also represented by the normal error curve the final distribution of temperature produced by these two different modes of transfer of heat must also be represented by an error curve. It can be proved by direct integration or otherwise that the mean of the square of deviations due to two independent causes of variation is equal to the sum of the means of the squares of deviations due to each separate cause. In the present problem this can be expressed by equation

$$\alpha^2 = \alpha^2_{\text{turb}} + \alpha^2_0, \tag{3}$$

which is identical with (1).

The temperature in the wake is represented by

$$\theta = \theta_0\, e^{-(Y^2/2\overline{Y^2})}, \tag{4}$$

where θ_0 is the rise in temperature in the middle. The half-breadth corresponding to $\theta = \frac{1}{2}\theta_0$ is therefore

$$Y_{\frac{1}{2}} = \sqrt{\overline{Y^2}}\sqrt{2\log_e 2} = 1{\cdot}177\, Y',$$

where $Y' = \sqrt{\overline{Y^2}}$.

In Schubauer's notation therefore

$$\alpha_{\text{turb}} = \frac{2Y_{\frac{1}{2}}}{x} = 2{\cdot}35\frac{Y'}{x}. \tag{5}$$

Hence, using (2), the theory of Part I leads to the expression

$$\alpha_{\text{turb}} = 2{\cdot}35\, v'/U, \tag{6}$$

so that α_{turb} is independent of x and depends only on v'/U. This is what in fact Schubauer finds experimentally.

Referring to fig. 5 (p. 4) of Schubauer's paper, he finds experimentally a linear connection between α_{turb} and the percentage turbulence u'/U which is measured by a hot wire. His results are best represented by $\alpha^0_{\text{turb}} = \frac{6}{0{\cdot}04}\frac{u'}{U}$, or, in circular measure,

$$\alpha_{\text{turb}} = 2{\cdot}62\, u'/U. \tag{7}$$

Comparing (7) with (6) it will be seen that they are identical if

$$\frac{v'}{\mathrm{U}} = \frac{2\cdot 62}{2\cdot 35}\frac{u'}{\mathrm{U}} = 1\cdot 11\,\frac{u'}{\mathrm{U}}. \tag{8}$$

It appears therefore that Schubauer's experimental result, when analysed according to the theory of Part I, leads to the conclusion that

$$v' = 1\cdot 11\, u'. \tag{9}$$

This is very nearly what would be found if the turbulence were in an isotropic condition for which $\overline{u^2} = \overline{v^2}$. It will be seen later that in experiments carried out at the National Physical Laboratory values of v'/U *less* than those of u'/U were sometimes obtained. Since Schubauer's experimental results show that v'/U is proportional to u'/U, and since some sets of measurements give $u' > v'$ while others give $u' < v'$, but all give values of the two quantities which are nearly equal, since, moreover, the measurements of u'/U depend on highly complicated systems of amplification, and are therefore susceptible to errors which are difficult to estimate, it seems legitimate to conclude that the turbulent flow produced by a grid becomes practically *isotropic* at some distance behind it. Alternatively isotropy might be assumed to exist; the analysis of Part I would then predict *a priori* Schubauer's experimental results with an error of only 11%.

Measurements of Diffusion over an Extended Range

The measurements described by Schubauer were made at distances from ½ to 6 inches from the heated source. These distances are not sufficiently great for the effect of the predicted decrease in R_ξ to be apparent, assuming it to exist. Recently Simmons has made measurements behind a heated flat strip and a heated wire in a wind tunnel at distances downstream varying from 2 to 36 inches. For this purpose he had to use a very intense source of considerable length (8 inches). These experiments are not yet complete, but one set of measurements behind a 0·9-inch honeycomb have been completed, and it is of interest to analyse them in the light of the theory of Part I. The measurements were made at distances x = 2, 4·2, 5·2, 6, 9, 12, 18, 24, 30, and 34 inches behind a nichrome strip raised to a very high temperature which was stretched across a 1-foot wind tunnel 23 inches behind a honeycomb with square cells 0·9 inch by 0·9 inch.

Some typical temperature distributions are shown in figs. 1 and 2. That at x = 2 inches shown in fig. 1 is similar to normal error distribution.

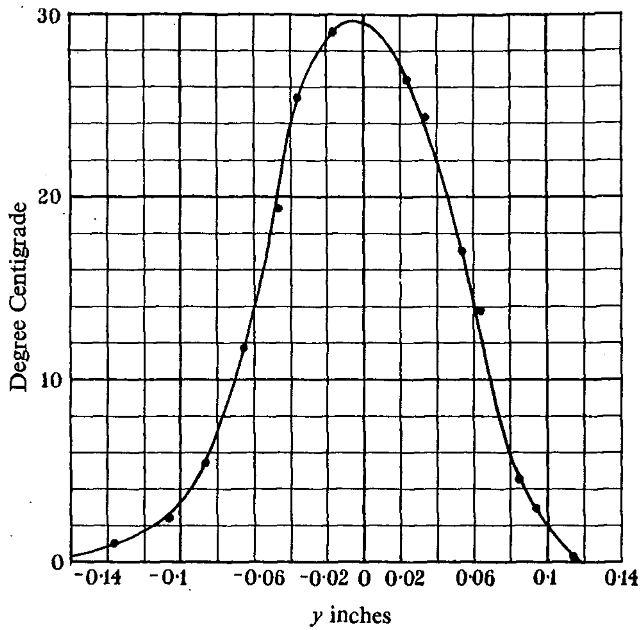


FIG. 1—Temperature of heat water 2 inches behind heated strip

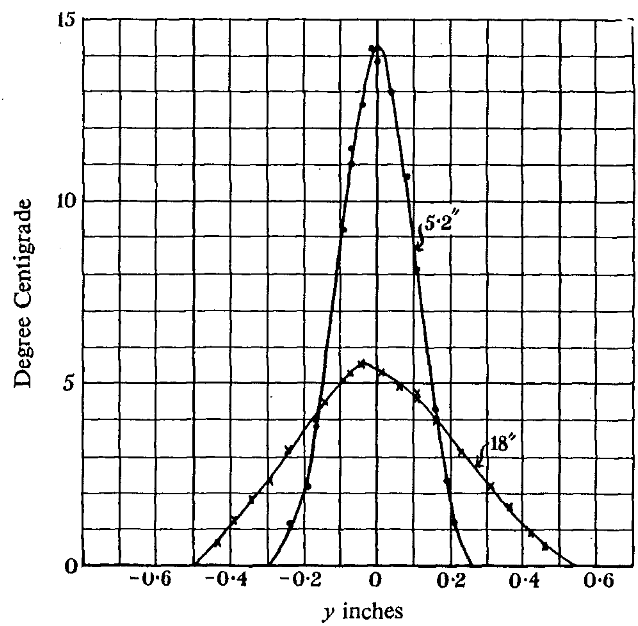


FIG. 2—Temperature of heat water 5·2 inches and 18 inches behind heated strip

The distribution at $x = 5 \cdot 2$ inches and 18 inches shown in fig. 2 appear to differ considerably from the normal error curve, being more pointed at the top and having a more definite outer edge. Each of the measured temperature distributions was analysed as follows: first a centre was chosen so that the temperatures were as equal as possible at equal distance on the two sides of the centre. If y is the distance from this centre, corresponding values of θ and y were read off from the observed distributions and values of $\dfrac{\int y^2\theta\, dy}{\int \theta\, dy}$ were found by direct summation. Representing these by Y'^2 the values of Y' so obtained are given in column 2, Table I.

TABLE I

x inches	Y' observed inches	Y' inches
0	0	0
2	0·0465	0·0409
4·2	0·0828	0·0764
5·2	0·0935	0·0916
6	0·0967*	0·0888*
9	0·120*	0·1104*
12	0·163	0·154
18	0·205	0·194
24	0·244	0·231
30	0·270	0·255
36	0·302	0·287
36	0·274*	0·257*

* Earlier observations.

Since the observations were taken on different days under different weather conditions in an open-ended wind tunnel it was to be expected that the turbulence conditions would not be constant during the whole course of the experiments. Some of the earlier results seem to have been obtained under conditions of less turbulence than the later ones. They are starred in Table I, and the corresponding points in fig. 3 are marked with crosses.

To find what the spread of the wake would have been if there had been no thermal conductivity, Schubauer's expression (1) may be used. In the present case it may be written

$$Y'^2 = (Y'_{obs})^2 - Y^2_0, \tag{10}$$

where Y_0^2 is the value which would be found for $\dfrac{\int y^2\theta\, dy}{\int \theta\, dy}$ in a wake spreading under the action of conductivity alone in a non-turbulent stream. If κ is the thermal conductivity, ρ density, σ specific heat of air, then

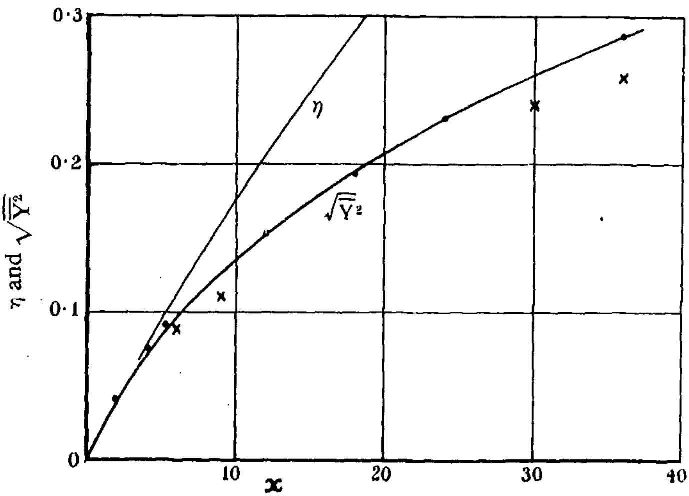
$$Y_0^2 = \frac{2\kappa x}{\rho\sigma U}, \tag{11}$$

and when $U = 20$ ft per sec, $\kappa = 5{\cdot}6 \times 10^{-5}$ c.g.s., $\sigma = 0{\cdot}24$

$$Y_0^2 = (2{\cdot}45 \times 10^{-4})\, x$$

square inches where x is expressed in inches.

Using (10) and (11) the values of Y' given in column 3, Table I, were calculated. These are set forth in fig. 3. To apply the analysis of Part I it is necessary to draw a smooth curve as near as possible to the observed points in fig. 3. This has been done; but it will be noticed that con-



x inches

FIG. 3

siderable latitude in drawing the curve is possible so that the results ultimately obtained are subject to considerable error and must be treated with reserve till experiments can be carried out under better experimental conditions.

ANALYSIS OF (Y′, x) CURVE, FIG. 3

To complete the analysis on the lines laid down in Part I we must know U/v' as a function of x. It has been shown in Part II that U/u' is a linear function of x, and that

$$\frac{U}{u'} = \frac{5}{A^2}\frac{x}{M} + \text{const.} \tag{12}$$

Measurements made in the stream behind the 0·9-inch by 0·9-inch honeycomb showed that $U/u' = \text{const.} + 1{\cdot}30x/M$, and when the measurements of u' were made U/u' was found to be 45 at the position where the source of heat in the present experiments was placed ($x = 0$ in these experiments). On the other hand, it was shown in Part I that the initial value of dY'/dx must be equal to v'/U at the source, and using the faired curve of fig. 3 the tangent at the original is consistent with $v'/U = 0{\cdot}02$ or $U/v' = 50$. Thus according to these measurements

$$\left[\frac{U}{u'}\right]_{x=0} = 45 \qquad \text{and} \left[\frac{U}{v'}\right]_{x=0} = 50,$$

or u' is 10% greater than v' at $x = 0$. That they are approximately equal lends further support to the assumption that turbulence behind honeycombs settles down to an isotropic state.

If the turbulence is isotropic we may therefore take

$$U/v' = [U/v']_{x=0} + 1{\cdot}30\frac{x}{M} = \text{constant} + 1{\cdot}44\,x, \tag{13}$$

where x is expressed in inches. It has been mentioned above that $[U/v']_{x=0} = 50$, so that (13) becomes

$$U/v' = 50 + 1{\cdot}44\,x. \tag{14}$$

By taking tangents to the faired curve, fig. 3, the values of Y′ and dY'/dx given in columns 3 and 4 of Table II were found. The values of appropriate values of U/v' from (14) are given in column 5, and in column 6 are the values of $\frac{U}{v'}Y'\frac{dY'}{dx}$.

CALCULATION OF η

It has been shown in Part I that

$$\frac{U}{v'}Y'\frac{dY'}{dx} = \int_0^{\eta} R_\eta\, d\eta$$

* Compare Part II, Table IV (p. 452).

should be a function of η where $\eta = \int_0^x \frac{v'}{U} dx$. Using the expression (14) for v'/U

$$\eta = \int_0^x \frac{dx}{50 + 1\cdot 44\, x} = \frac{1}{1\cdot 44} \log_e (1 + 0\cdot 0288\, x). \qquad (15)$$

The values of η calculated from (15) are given in column 2 of Table II and in fig. 3 the values of η are shown. The difference between the η curve and the faired curve for Y′ represents the effect of the decrease in R_η as η increases. Near $x = 0$ where $R_\eta = 1\cdot 0$ the η curve coincides with the Y′ curve.

TABLE II

1	2	3	4	5	6	7
x	η inches	Y′ inches	dY'/dx	U/v'	$\frac{U}{v'} Y' \frac{dY'}{dx}$	η/M
0	0	0	0·020	50	0	0
2	0·039	0·039	0·0374	52·9	0·0359	0·043
4 or	0·075	0·074	0·0135	55·7	0·0555	0·084
6	0·111	0·096	0·0115	58·6	0·0648	0·123
8	0·144	0·118	0·0100	61·5	0·0726	0·160
12	0·206	0·154	0·0080	67·3	0·083	0·229
18	0·290	0·196	0·0063	75·9	0·093	0·322
24	0·365	0·231	0·0054	84·5	0·105	0·405
30	0·432	0·284	0·0044	93·1	0·108	0·480
36	0·493	0·284	{0·0036, 0·0040}	101·7	{0·104, 0·115}	0·547

CALCULATION OF R_η

In fig. 4 are shown the values of $\frac{U}{v'} Y' \frac{dY'}{dx}$, taken from Table II, as ordinates, with η as abscissae. It will be seen that $\frac{U}{v'} Y' \frac{dY'}{dx}$ increase rapidly at first and then more slowly, but it is impossible to tell from these observations whether the curve is still sloping upwards at the greatest value of x or not.

It was shown in Part I that $R_\eta = \frac{d}{d\eta}\left(\frac{U}{v'} Y' \frac{dY'}{dx}\right)$. The values of the R_η obtained in this way by graphical differentiation of the curve of fig. 4 are given in Table III and also shown in fig. 4. From this curve it appears that R_η falls very rapidly at first but more slowly later, so that there is still a small correlation equal to 0·05 at $\eta = 0\cdot 4$ inch. Since this result depends very much on the manner in which the faired curve for Y′ is obtained from the observations it must be treated with reserve. The values of η/M are given at the bottom of Table III, because it may be expected that when turbulence of varying scales are compared, R_η will be a function of η/M.

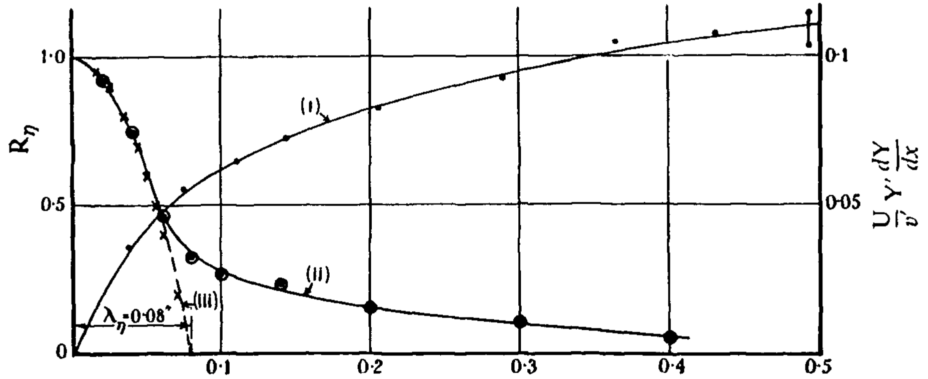


FIG. 4—(i) • $\frac{U}{v'} Y' \frac{dY'}{dx}$; (ii) ⊙ $R_\eta = \frac{d}{dr_i}\left\{\frac{U}{v'} Y' \frac{dY'}{dx}\right\}$; (iii) × parabola $R_y = 1 - \frac{r_i^2}{(0{\cdot}08)^2}$

TABLE III

η	0·02	0·04	0·06	0·08	0·10	0·14	0·20	0·30	0·40
R_η	0·92	0·75	0·47	0·33	0·27	0·24	0·16	0·11	0·056
η/M	0·022	0·044	0·067	0·089	0·111	0·155	0·222	0·333	0·444

VALUE OF l_1

If $\frac{U}{v'} Y' \frac{dY'}{dx}$ has attained the maximum value of 0·11 at $r_i = 0''{\cdot}5$ inch then according to equation (15), Part I, this is equal to l_1, so that

$$\frac{l_1}{M} = \frac{0{\cdot}11}{0{\cdot}9} = 0{\cdot}12. \qquad (16)$$

CURVATURE OF R_η CURVE AT ORIGIN

On examining the R_η curve it appeared to be nearly parabolic near the origin.

It has been shown that the R_y curve in the Eulerian representation of turbulent flow is parabolic near the origin, and the significance of this in relation to dissipation is discussed in Parts I, II, and III. It is of interest therefore to find out how far the R_η curve is also parabolic near the origin. In fig. 4 the parabola $R_y = 1 - \left(\frac{\eta}{0 \cdot 08}\right)^2$ is drawn. It will be seen that it coincides with the observed R_η curve over a considerable range.

SIGNIFICANCE OF THE CURVATURE OF THE R_η CURVE AT $\eta = 0$

The form of correlation curves near the origin is discussed in Part I. In the present case the general expression takes the form

$$\overline{\left(\frac{Dv}{D\eta}\right)^2} = 2v'^2 \, \mathrm{Lt}_{\eta \to 0} \left(\frac{1 - R_\eta}{\eta^2}\right), \tag{17}$$

where $Dv/D\eta$ is the rate of change with η of the v component of velocity of a particle as it moves down-stream with velocity U. Since η is really a length depending essentially on time in the sense that

$$\delta\eta = \frac{v'}{U} \, \delta x = v' \, \delta t,$$

therefore

$$\overline{\left(\frac{Dv}{D\eta}\right)^2} = \frac{1}{v'^2} \overline{\left(\frac{Dv}{Dt}\right)^2}. \tag{18}$$

Hence (17) may be written

$$\overline{\left(\frac{Dv}{Dt}\right)^2} = 2v'^4 \, \mathrm{Lt} \left(\frac{1 - R_\eta}{\eta^2}\right). \tag{19}$$

Bearing in mind the method of representing the curvature of the R_y curves, which is explained in Part I, we may define the length λ_η so that

$$\frac{1}{\lambda_\eta^2} = \mathop{\mathrm{Lt}}_{\eta \to 0} \left(\frac{1 - R_\eta}{\eta^2}\right), \tag{20}$$

λ_η is then the intercept on the axis $R_\eta = 0$ of the parabola which touches the R_η curve at $\eta = 0$. In the case represented in fig. 4 $\lambda_\eta = 0 \cdot 08$ inches. (19) may be written

$$\overline{\left(\frac{Dv}{Dt}\right)^2} = 2 \frac{v'^4}{\lambda_\eta^2}. \tag{21}$$

Now Dv/Dt is simply the acceleration of a particle in the direction y expressed in the Lagrangian conception of fluid kinematics. The Lagrangian equation of motion in the y direction is

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{Dv}{Dt}. \tag{22}$$

Hence when the dynamical equations are considered it will be seen that

$$\overline{\left(\frac{\partial p}{\partial y}\right)^2} = 2\rho^2 \frac{v'^4}{\lambda_\eta^2}. \tag{23}$$

It appears therefore that it should be possible from experiments on diffusion to find the mean rate of change in pressure in turbulent motion, namely $\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$.

This seems to be important because the disturbing effect of turbulence on the laminar boundary layer at the surface of a solid moving in a stream of fluid might be ascribed to the pressure gradients which accompany turbulent motion. If that assumption is made it seems that it is now possible to express this disturbing effect in a quantitative manner by means of the Karman-Pohlhausen boundary layer equations, using the results of applying (23) to diffusion experiments. Considerable progress has already been made along these lines, a report of which will, it is hoped, be published shortly.

Relationship between λ and $\lambda\eta$

The mean spatial rate of change in pressure $\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$ in turbulent motion has apparently never been discussed theoretically, nor have any observations been made on it. Indeed, no observations, even of the pressure variation $\sqrt{\overline{p^2}}$, appear to have been published, though some were made many years ago by Pannell* but were never published. It seems certain that $\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$ must depend on $\sqrt{\overline{p^2}}$ and on the mean spatial rate of change in velocity $\sqrt{\overline{\left(\frac{\partial v}{\partial y}\right)^2}}$.

The equation of motion in the Eulerian system is

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} + \tfrac{1}{2}\frac{\partial}{\partial y}(u^2 + v^2 + w^2) - w\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + u\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{24}$$

It seems to be impossible, without making further assumptions about the nature of turbulent motion, to assign even an approximate value to the terms on the right-hand side of (24). It seems likely that $\frac{1}{\rho}\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$

* 'Rep. Mem. Adv. Cttee. Aero.,' No. 345 (August, 1917).

will be of the same order of magnitude as $\frac{1}{2}\sqrt{\overline{\left[\frac{\partial}{\partial y}(u^2+v^2+w^2)\right]^2}}$, which in isotropic turbulence is likely to be of order $\frac{3}{2}\rho\sqrt{\overline{\left(\frac{\partial v^2}{\partial y}\right)^2}}$. The value of $\overline{\left(\frac{\partial v^2}{\partial y}\right)^2}$ or $\overline{4v^2\left(\frac{\partial v}{\partial y}\right)^2}$ must be of the same order of magnitude as $4\,\overline{v^2}\,\overline{\left(\frac{\partial v}{\partial y}\right)^2}$; but the exact relationship will depend on the frequency distribution of v and $\partial v/\partial y$. It seems, therefore, that $\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$ is likely to be of the same order of magnitude as $3\rho\sqrt{\overline{v^2}\,\overline{\left(\frac{\partial v}{\partial y}\right)^2}}$.

Assumption—Let us therefore assume that in isotropic turbulence

$$\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}} = 3\mathrm{B}\rho\sqrt{\overline{v^2}\,\overline{\left(\frac{\partial v}{\partial y}\right)^2}}, \tag{25}$$

where B is a constant which is expected to be of order of magnitude unity.

From (23) and (25)

$$\mathrm{B}^2 = \frac{2v'^4}{9\lambda_\eta^2\,\overline{v^2}\,\overline{\left(\frac{\partial v}{\partial y}\right)^2}}. \tag{26}$$

The value of $\overline{\left(\frac{\partial v}{\partial y}\right)^2}$ is discussed in Parts I and II. In isotropic* turbulence it is $\frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ and †$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \frac{2\overline{u^2}}{\lambda^2}$ so that $\overline{\left(\frac{\partial v}{\partial y}\right)^2} = \frac{\overline{v^2}}{\lambda^2}$.

Hence

$$\mathrm{B}^2 = \frac{2}{9}\left(\frac{\lambda^2}{\lambda_\eta^2}\right). \tag{27}$$

The assumption which is represented by (25) is therefore equivalent to the assumption that λ_η *is constant multiple of* λ.

COMPARISON WITH OBSERVATION

For the turbulent air behind a square-mesh honeycomb, M = 0·9 inch, it was found that U = 20 ft per sec, λ_η = 0·08 inch (*see* fig. 4).

* Equation (44), Part I.

† Equations (47) and (48), Part I.

For the same honeycomb at $U = 25$ ft per sec it was shown that direct observation and calculation from the observed energy dissipation agree in giving $\lambda = 0\cdot143$ inch.

Since for a given size of honeycomb λ is proportional to* $u'^{\frac{1}{2}}$ and since within wide limits u'/U is constant for a fixed distance behind the honeycomb as U varies, the value of λ at $U = 20$ ft per sec is

$$\lambda = 0\cdot143\sqrt{\frac{25}{20}} = 0\cdot16 \text{ inch.} \tag{28}$$

Hence from (27)

$$B = \sqrt{\frac{2}{9}}\left(\frac{0\cdot16}{0\cdot08}\right) = 0\cdot94. \tag{29}$$

The fact that B happens to be so nearly equal to $1\cdot0$ can have no significance in view of the uncertainty in determining the R_η curve, figs. 3 and 4, but the fact that B turns out to be of the order of magnitude unity does seem to confirm the general accuracy of the conception here put forward of the connection between the statistical representation of turbulent flow in the Eulerian and Lagrangian conceptions. The whole question clearly needs more experimental work on the lines laid down in the present series of papers.

In conclusion I should like to express my thanks to Mr. F. L. G. Simmons of the National Physical Laboratory for permission to make use of the unpublished data contained in figs. 1 and 2 and in Table I.

* Equation (56), Part I.

On the Statistical Theory of Isotropic Turbulence

By Theodore de Kármán and Leslie Howarth*

California Institute of Technology, Pasadena

(*Communicated by G. I. Taylor, F.R.S.—Received* 27 *January* 1937—
Revised 6 *November* 1937)

Introduction and Summary

G. I. Taylor (1935), in a paper of fundamental importance, introduced the conception of isotropic turbulence and applied it, with interesting results, to the problem of the decay of turbulence in a windstream.

In this paper we develop a general theory of isotropic turbulence. The correlation coefficients between two arbitrary velocity components at two arbitrary points form a tensor called the correlation tensor. Due to isotropy the tensor corresponding to one fixed and one variable point has spherical symmetry; due to the condition of continuity it is completely determined by one scalar function. The mean products of the derivatives of the velocity fluctuations are expressed by the derivatives of the tensor components; in this way laborious calculations for obtaining such mean values are eliminated. The correlation between three components (triple correlation) is discussed.

After developing the kinematics of isotropic turbulence the dynamical problem of the change of the various mean values with time is considered. It is shown that using the equations of motion a partial differential equation connecting the double and triple correlation functions can be established. The solution of this equation is investigated first in the case when the triple correlation is neglected and the shape of the double correlation function remains similar and only its scale changes. Equations for the dissipation of energy and vorticity are deduced.

Finally, a possible solution for large Reynolds numbers is given and applied to Taylor's problem of the decay of turbulence behind a grid.

Taylor's fundamental relation between the width of the correlation function and the size of the small (dissipative) eddies is confirmed. However, it is believed that the linear law for the reciprocal of the root mean square of

* Research Fellow, King's College, Cambridge.

the velocity fluctuations as a function of the distance from the grid is a special case; in the last section of the present paper a more general functional relation is suggested.

§§ 6, 8 and 9 have been rewritten and § 11 inserted by the senior author in September 1937.

A—The kinematics of isotropic turbulence

1—*The definition of isotropic turbulence and immediate deductions*

Isotropic turbulence may be defined by the condition that the average value of any function of the velocity components and their derivatives at a particular point, defined in relation to a particular set of axes, is unaltered if the axes of reference are rotated in any manner and if the co-ordinate system is reflected in any plane through the origin. We consider average values with regard to the time and suppose that the fluctuations are so rapid that the variation of the average value is negligible throughout the period of time required for averaging. Thus, in fact, we shall consider the average values to be slowly varying functions of the time.

Consider a particular system of co-ordinate axes Ox_1, Ox_2 and Ox_3 and take two points $P(x_1, 0, 0)$ and $P'(x'_1, 0, 0)$ on Ox_1. Denote by u_1, u_2, u_3 and u'_1, u'_2, u'_3 the velocity components at P and P' respectively. Let us now suppose that $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$ are all independent of position. (By isotropy, of course, $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$ ($=\overline{u^2}$, say).)

The correlation coefficients $\overline{u_1 u'_1}/\overline{u^2}$ and $\overline{u_j u'_j}/\overline{u^2}$ for $j = 2$ or 3 will be particular functions $f(r, t)$ and $g(r, t)$, say, of the distance r between P and P'* and of the time. It seems to be fairly evident physically that the mean value $\overline{u_i u'_j} = 0$ when $i \neq j$ in isotropic turbulence, since it appears to be equally probable that $u_1 u'_2$, for example, will be positive or negative. It is an easy matter to prove these results analytically. The mean values $\overline{u_1 u'_j}$ and $\overline{u'_1 u_j}$ for $j = 2$ or 3 can be shown to vanish by a rotation of the axes of x_2 and x_3 through 180° about the x_1-axis. Denoting transformed values by capital letters we see that $U_1 = u_1$, $U'_1 = u'_1$, $U_j = -u_j$ and $U'_j = -u'_j$ for $j = 2$ or 3 so that, for example,

$$\overline{U_1 U'_j} = -\overline{u_1 u'_j}.$$

But by the isotropic property

$$\overline{U_1 U'_j} = \overline{u_1 u'_j},$$

* It is assumed that the turbulence is statistically uniform as well as isotropic so that correlations do not depend on the position or orientation of the line PP' but only on its length.

so that $\overline{u_1 u_j'}$ is zero for $j = 2$ or 3. Similarly $\overline{u_j u_1'}$ is also zero for $j = 2$ or 3. The mean values $\overline{u_2 u_3'}$ and $\overline{u_3 u_2'}$ can be shown to vanish by reflexion in the x_1, x_3 plane. Denoting, in this case, reflected values by capital letters we see that $U_2 = -u_2$, $U_2' = -u_2'$, $U_3 = u_3$, $U_3' = u_3'$, so that, for example,

$$\overline{U_2 U_3'} = -\overline{u_2 u_3'}.$$

But by the isotropic property

$$\overline{U_2 U_3'} = \overline{u_2 u_3'},$$

and hence $\overline{u_2 u_3'}$ vanishes. Similarly $\overline{u_3 u_2'}$ is zero.

2—*Lemma*

Consider now any two points P and Q in the fluid. Denote by p the velocity component in a particular direction PP' at P and by q the velocity component in the direction QQ' at Q. We shall determine an expression for the correlation coefficient $\overline{pq}/\overline{u^2}$.

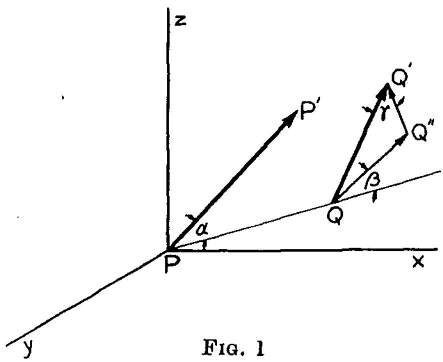


FIG. 1

The intersecting lines PP' and PQ (see fig. 1) determine a plane. Denote by QQ'' the orthogonal projection of QQ' on this plane; $Q''Q'$ will then be normal to the plane. Denote by α, β and γ the angles $P'\hat{P}Q$, $\pi - P\hat{Q}Q''$, $Q\hat{Q}'Q''$, and by p_1, p_2, p_3 and q_1, q_2, q_3, respectively, the velocity components at P and Q in the direction PQ, the direction normal to PQ and $Q''Q'$, and the direction $Q''Q'$.* Then

$$p = p_1 \cos\alpha + p_2 \sin\alpha,$$

$$q = q_1 \cos\beta \sin\gamma + q_2 \sin\beta \sin\gamma + q_3 \cos\gamma.$$

* For simplicity the figure is described as it is drawn in fig. 1. When P' and Q'' are on opposite sides of PQ the sign of β must be changed.

Therefore

$$\overline{pq} = \overline{p_1q_1}\cos\alpha\,\cos\beta\,\sin\gamma + \overline{p_1q_2}\cos\alpha\,\sin\beta\,\sin\gamma + \overline{p_1q_3}\cos\alpha\,\cos\gamma$$
$$+\overline{p_2q_1}\cos\beta\,\sin\gamma\,\sin\alpha + \overline{p_2q_2}\sin\beta\,\sin\alpha\,\sin\gamma + \overline{p_2q_3}\sin\alpha\,\cos\gamma.$$

We have already proved that all the mean values here except $\overline{p_1q_1}$ and $\overline{p_2q_2}$ vanish, and we have denoted by $f(r, t)$ and $g(r, t)$ the correlation coefficients $\overline{p_1q_1}/\overline{u^2}$ and $\overline{p_2q_2}/\overline{u^2}$, where PQ is supposed to be of length r. Thus

$$\frac{\overline{pq}}{\overline{u^2}} = [f(r,t)\cos\alpha\cos\beta + g(r,t)\sin\alpha\sin\beta]\sin\gamma. \tag{1}$$

3—*The correlation tensor*

Consider now any particular co-ordinate system and suppose the velocity components at $P(x_1, x_2, x_3)$ and $P'(x_1', x_2', x_3')$ are (u_1, u_2, u_3) and (u_1', u_2', u_3') respectively. The nine quantities $\overline{u_iu_j'}$ for $i, j = 1$, 2 or 3 may be shown to be the components of a second rank tensor. The transformation law can readily be seen to be satisfied since the dyadic product of two vectors is a tensor so that each of the contributions u_iu_j' to the mean values form a tensor. Clearly the operation of taking a mean value will not alter the transformation law satisfied. Hence we may consider the "correlation tensor" **R** defined by

$$\overline{u^2}\mathbf{R} = \overline{u^2}\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} \overline{u_1u_1'} & \overline{u_1u_2'} & \overline{u_1u_3'} \\ \overline{u_2u_1'} & \overline{u_2u_2'} & \overline{u_2u_3'} \\ \overline{u_3u_1'} & \overline{u_3u_2'} & \overline{u_3u_3'} \end{pmatrix}$$

or

$$\overline{u^2}\mathbf{R} = \overline{u^2}R_{ij} = \overline{u_iu_j'} \qquad i,j = 1,\ 2 \text{ or } 3. \tag{2}$$

By means of the lemma which has been established each component of **R** can be evaluated in terms of the functions f and g and the vector $\mathbf{r}$ whose components are $\xi_1 = x_1' - x_1$, $\xi_2 = x_2' - x_2$, $\xi_3 = x_3' - x_3$. In this way we obtain

$$\mathbf{R} = \frac{\{f(r,t) - g(r,t)\}}{r^2}\mathbf{rr} + g(r,t)\,\mathbf{I}, \tag{3}$$

where $r = |\mathbf{r}|$ and $\mathbf{I}$ is the unit tensor $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By expressing the fact that the velocity fluctuations satisfy the equation of continuity we can now obtain a relation between $f(r, t)$ and $g(r, t)$. Since the velocity fluctuations at P' satisfy the equation of continuity

$$\frac{\partial u_i'}{\partial x_i'} = 0,$$

where the summation convention is in operation. Therefore

$$\frac{\partial}{\partial x_i'}(u_i' u_j) = 0, \tag{4}$$

since u_j is independent of x_1', x_2' and x_3'. Thus we obtain the result

$$\frac{\partial R_{ji}}{\partial x_i'} = 0, \tag{5}$$

for $j = 1$, 2 or 3 by dividing equation (4) by $\overline{u^2}$ (which is independent of position), by taking mean values and using the definition of R_{ji} (see equation (2)). By the definition of ξ_1, ξ_2, ξ_3 we may write equation (5) as

$$\frac{\partial R_{ji}}{\partial \xi_i} = 0. \tag{6}$$

Using the values of the components of $\mathbf{R}$ given by equation (3), and also the fact that equation (6) is valid for all values of ξ_1, ξ_2 and ξ_3, we find

$$2f(r, t) - 2g(r, t) = -r\frac{\partial f(r, t)}{\partial r}. \tag{7}$$

Since both $f(r, t)$ and $g(r, t)$ are even functions of r

$$f(r, t) = 1 + f_0''\frac{r^2}{2!} + f_0^{\text{iv}}\frac{r^4}{4!} + \dots, \tag{8}$$

$$g(r, t) = 1 + g_0''\frac{r^2}{2!} + g_0^{\text{iv}}\frac{r^4}{4!} + \dots. \tag{9}$$

From equation (7) we then find by equating coefficients of r^2 and r^4

$$2f_0'' = g_0'', \tag{10}$$

$$3f_0^{\text{iv}} = g_0^{\text{iv}}. \tag{11}$$

Further, substituting these expansions for f and g in equation (3) we find, for small values of r, that

$$\mathbf{R} = \left(1 + \frac{g_0''}{2}r^2\right)\mathbf{I} + \left(\frac{f_0'' - g_0''}{2}\right)\mathbf{rr}, \tag{12}$$

neglecting fourth and higher powers of r. Using equation (10) we may write equation (12) in the form

$$\mathbf{R} = (1 + f''_0 r^2)\,\mathbf{I} - \tfrac{1}{2} f''_0\, \mathbf{rr}. \tag{13}$$

We notice, in passing, that

$$\left.\begin{aligned} \left(\frac{\partial^2 R_{kl}}{\partial \xi_i\, \partial \xi_j}\right)_{\xi_1=\xi_2=\xi_3=0} &= f''_0 \quad \text{when} \quad k = l = i = j, \\ &= 2f''_0 \quad \text{when} \quad k = l \neq i = j, \\ &= -\tfrac{1}{2} f''_0 \quad \text{when} \quad k = i \neq l = j \quad \text{or} \quad k = j \neq l = i, \\ &= 0 \text{ otherwise;} \end{aligned}\right\} \tag{14}$$

we shall require these results later.

We may, at this stage, point out an analogy which is helpful in creating a physical picture. The expression (3) for the correlation tensor is of exactly the same form as that for the stress tensor for a continuous medium when there is spherical symmetry. In the analogy $f(r)$ is the principal radial stress at any point, $g(r)$ is any of the principal transverse stresses, and R_{ik} is the component in the k direction of the stress over a plane whose normal is in the i direction. Further, the relation between f and g given by continuity in our problem corresponds to the condition for equilibrium in the stress analogy.

4—*The correlation coefficients between derivatives of the velocities*

Taylor (1935) had occasion to calculate the various correlation coefficients between first derivatives of the velocity components. These calculations were made by transforming axes, using the equation of continuity and applying the definition of isotropy. Even for the first derivatives this method is fairly laborious, whilst for higher order derivatives the work involved makes it almost prohibitive; it is hardly possible to decide beforehand whether a particular transformation will lead to a new relation or will merely give one that has already been obtained.

The determination of the correlations both quickly and automatically is an interesting application of the correlation tensor. Let us take, as an example, the determination of $\overline{\frac{\partial u_k}{\partial x_i}\frac{\partial u_l}{\partial x_j}}$. Now

$$\frac{\partial}{\partial x_i}\overline{(u_k u'_l)} = \overline{u^2}\frac{\partial}{\partial x_i} R_{kl} = -\overline{u^2}\frac{\partial R_{kl}}{\partial \xi_i}, \tag{15}$$

i.e. since the velocity fluctuations at P are all differentiable functions of x_i,

$$\overline{\frac{\partial u_k}{\partial x_i} u_l'} = -\overline{u^2} \frac{\partial R_{kl}}{\partial \xi_i}. \tag{16}$$

Further
$$\frac{\partial}{\partial x_j'} \left(\overline{\frac{\partial u_k}{\partial x_i} u_l'} \right) = -\overline{u^2} \frac{\partial}{\partial x_j'} \left(\frac{\partial R_{kl}}{\partial \xi_i} \right) = -\overline{u^2} \frac{\partial^2 R_{kl}}{\partial \xi_j \, \partial \xi_i},$$

i.e.
$$\overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_l'}{\partial x_j'}} = -\overline{u^2} \frac{\partial^2 R_{kl}}{\partial \xi_i \, \partial \xi_j}. \tag{17}$$

If, now, we make P and P' coincide we obtain the result

$$\overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_j}} = -\overline{u^2} \left(\frac{\partial^2 R_{kl}}{\partial \xi_i \, \partial \xi_j} \right)_{\xi_1 = \xi_2 = \xi_3 = 0} \tag{18}$$

We obtain the values of $\dfrac{\partial^2 R_{kl}}{\partial \xi_i \, \partial \xi_j}$ when $\xi_1 = \xi_2 = \xi_3 = 0$ immediately from equations (14). Hence, for example, we see that

$$\overline{\left(\frac{\partial u_1}{\partial x_1} \right)^2} = -\overline{u^2} f_0'', \tag{19}$$

$$\overline{\left(\frac{\partial u_1}{\partial x_2} \right)^2} = -2\overline{u^2} f_0'', \tag{20}$$

and that
$$\overline{\frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2}} = \overline{\frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1}} = \tfrac{1}{2} \overline{u^2} f_0''$$
$$= -\frac{1}{2} \overline{\left(\frac{\partial u_1}{\partial x_1} \right)^2}, \tag{21}$$

or, alternatively,
$$= -\frac{1}{4} \overline{\left(\frac{\partial u_1}{\partial x_2} \right)^2}, \tag{22}$$

by equations (19) and (20) respectively.

Similarly, all other correlations of first derivatives can be obtained.

This method is immediately applicable to higher order derivatives. Exactly as for the first derivatives we obtain, for example, the result

$$\overline{\frac{\partial^2 u_k}{\partial x_i \, \partial x_j} \frac{\partial^2 u_l}{\partial x_r \, \partial x_s}} = \overline{u^2} \left(\frac{\partial^4 R_{kl}}{\partial \xi_i \partial \xi_j \, \partial \xi_r \, \partial \xi_s} \right)_{\xi_1 = \xi_2 = \xi_3 = 0}. \tag{23}$$

The fourth derivative of the components of **R** can be obtained from equation (3), and using the expansions (8) and (9) we find, for example,

$$\overline{\left(\frac{\partial^2 u_1}{\partial x_1^2} \right)^2} = \overline{u^2} f_0^{\text{iv}}, \tag{24}$$

$$\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2} = 3\overline{u^2} f_0^{\text{iv}}, \tag{25}$$

and
$$\overline{u^2}\left(\frac{\partial^4 R_{11}}{\partial \xi_1^2 \partial \xi_2^2}\right)_{\xi_1=\xi_2=\xi_3=0} = \tfrac{2}{3}\overline{u^2} f_0^{\text{iv}},$$

so that
$$\overline{\frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_1}{\partial x_2^2}} = \overline{\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)^2} = \frac{2}{9}\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2} = \frac{2}{3}\overline{\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2}, \tag{26}$$

and so on.

For completeness we include the results for first and second derivatives, writing (u, v, w) and (x, y, z) for (u_1, u_2, u_3) and (x_1, x_2, x_3). For the first derivatives*

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}, \quad \overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = \overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} = -\frac{1}{4}\overline{\left(\frac{\partial u}{\partial y}\right)^2}. \tag{27}$$

For the second derivatives

$$\left.\begin{aligned}
&\overline{\left(\frac{\partial^2 u}{\partial x^2}\right)^2} = \overline{\left(\frac{\partial^2 u}{\partial y \partial z}\right)^2} = \overline{\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial z^2}} = \frac{1}{3}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2},\\
&\overline{\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2} = \overline{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}} = \frac{2}{9}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2},\\
&\overline{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x \partial y}} = \overline{\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x \partial y}} = -\frac{1}{6}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2},\\
&\overline{\frac{\partial^2 u}{\partial x \partial z}\frac{\partial^2 v}{\partial y \partial z}} = \overline{\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 v}{\partial x \partial y}} = \overline{\frac{\partial^2 u}{\partial y \partial z}\frac{\partial^2 v}{\partial z \partial x}} = -\frac{1}{18}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}.
\end{aligned}\right\} \tag{28}$$

All mean values which cannot be brought into one of these forms by cyclic permutation of the letters vanish.

It is obvious that the process could be continued and the higher derivatives of R_{ik} give analogous expressions for the mean products of the higher derivatives of the velocity components. Taylor's development formula for his correlation function R_y (1935, equation (46)) is a special case of the general formulae obtained in this way. His equation, which he effectively found as early as 1921, would be expressed in our notation

$$g(r, t) = 1 - \frac{\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}}{\overline{u^2}}\frac{r^2}{2!} + \frac{\overline{\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2}}{\overline{u^2}}\frac{r^4}{4!} - \dots.$$

An analogous expression can be written for $f(r, t)$.

* These results were first given by Taylor (1935, p. 436).

5—*Expression of mean values by integrals*

It is known that the velocity components in an incompressible fluid can be expressed by a vector and a scalar potential. In the present case (indefinitely extended fluid without sources throughout the whole fluid) the scalar potential vanishes and the velocity components can be written (it is convenient to use x, y, z, u, v, w for the co-ordinates and the velocity components in this section)

$$u = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \quad v = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \quad w = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}, \tag{29}*$$

where

$$F = \frac{1}{4\pi}\iiint \frac{\omega'_x}{r}\,dx'\,dy'\,dz', \quad G = \frac{1}{4\pi}\iiint \frac{\omega'_y}{r}\,dx'\,dy'\,dz',$$
$$H = \frac{1}{4\pi}\iiint \frac{\omega'_z}{r}\,dx'\,dy'\,dz', \tag{30}$$

ω'_x, ω'_y, ω'_z are the components of vorticity at the point (x', y', z'),

$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$

and the integration extends throughout the fluid. These results enable us to express correlations between vorticity components and velocity derivatives conveniently as integrals. Consider, for example, $\overline{\omega_x \frac{\partial w}{\partial y}}$. Now

$$\frac{\partial w}{\partial y} = \frac{1}{4\pi}\left\{\frac{\partial^2}{\partial y\,\partial x}\iiint \frac{\omega'_y}{r}\,dx'\,dy'\,dz' - \frac{\partial^2}{\partial y^2}\iiint \frac{\omega'_x}{r}\,dx'\,dy'\,dz'\right\}$$
$$= \frac{1}{4\pi}\iiint\left[\frac{3(x'-x)(y'-y)}{r^5}\,\omega'_y - \left\{\frac{3(y'-y)^2}{r^5} - \frac{1}{r^3}\right\}\omega'_x\right]dx'\,dy'\,dz', \tag{31}$$

so that

$$\overline{\omega_x \frac{\partial w}{\partial y}} = \frac{1}{4\pi}\iiint\left[\frac{3(x'-x)(y'-y)}{r^5}\,\overline{\omega_x\,\omega'_y} - \left\{\frac{3(y'-y)^2}{r^5} - \frac{1}{r^3}\right\}\overline{\omega_x\,\omega'_x}\right]dx'\,dy'\,dz'. \tag{32}$$

Now we can define a correlation tensor **V** for the vorticity components in precisely the same way as **R** was defined for the velocity components. Hence, using an obvious notation,

$$\overline{\omega_x\,\omega'_y} = \overline{\omega_x^2}\,V_{xy}, \quad \overline{\omega_x\,\omega'_x} = \overline{\omega_x^2}\,V_{xx}$$

* See Lamb (1932). These results are purely kinematic.

and

$$\overline{\omega_x \frac{\partial w}{\partial y}} = \frac{\overline{\omega_x^2}}{4\pi} \iiint \left[\frac{3(x'-x)(y'-y)}{r^5} V_{xy} - \left\{ \frac{3(y'-y)^2}{r^5} - \frac{1}{r^3} \right\} V_{xx} \right] dx'\,dy'\,dz'. \tag{33}$$

When the turbulence is isotropic this method of approach has no special merit since the required results may be obtained more easily by other methods. It is clear that the form for **V** is exactly similar to that for **R** (including the condition arising from continuity) in isotropic flow; using this fact the integral in equation (33) can be evaluated and the result compared with the one that can be deduced immediately from equations (27).

6—*Triple correlations**

We shall now consider the mean values of the product of three-velocity components u_i, u_j, u'_k, where u_i and u_j are the instantaneous values of the i and j components of the velocity observed at an arbitrary point P and u'_k is the k-component of the velocity observed at another arbitrary point P'. Let us consider again homogeneous isotropic turbulence. Then

$$\overline{u_i^2} = \overline{u_j^2} = \overline{u_k'^2} = \overline{u^2},$$

and we write

$$\overline{u_i u_j u'_k} = (\overline{u^2})^{3/2}\, T_{ijk},$$

where T_{ijk} is a tensor of third rank. The tensor T_{ijk} will be a function of ξ_1, ξ_2, ξ_3 where $\xi_1 = x'_1 - x_1$, $\xi_2 = x'_2 - x_2$, $\xi_3 = x'_3 - x_3$; we call it the tensor of the triple correlations. In order to find the expression for T_{ijk} as a function of the ξ's, let us assume first that both points P and P' lie on the x-axis. Then it is obvious that all quantities $\overline{u_i u_j u'_k}$ belong to one of the following six groups, $\overline{u_1^2 u'_1}$, $\overline{u_1^2 u'_k}$, $\overline{u_1 u_i u'_1}$, $\overline{u_1 u_j u'_k}$, $\overline{u_i u_j u'_1}$ and $\overline{u_i u_j u'_k}$, where i, j, k can be equal to 2 or 3. Due to the assumption of isotropy $\overline{u_1^2 u'_k} = 0$, $\overline{u_1 u_i u'_1} = 0$, and $\overline{u_i u_j u'_k} = 0$, because by reflexion of at least one of the axes x_2 or x_3 these expressions certainly change their sign. Furthermore, $\overline{u_1 u_j u'_k}$ and $\overline{u_i u_j u'_1}$ vanish for the same reason unless $j = k$ or $i = j$. Hence only the following mean values remain as possibly different from zero: $\overline{u_1^2 u'_1}$, $\overline{u_1 u_2 u'_2}$, $\overline{u_1 u_3 u'_3}$, $\overline{u_2^2 u'_1}$, $\overline{u_3^2 u'_1}$. Obviously because of isotropy $\overline{u_1 u_2 u'_2} = \overline{u_1 u_3 u'_3}$ and $\overline{u_2^2 u'_1} = \overline{u_3^2 u'_1}$. Hence three independent quantities remain different from zero; we put $\overline{u_1^2 u'_1} = (\overline{u^2})^{3/2}\, k(r)$, $\overline{u_1 u_2 u'_2} = \overline{u_1 u_3 u'_3} = (\overline{u^2})^{3/2}\, q(r)$ and $\overline{u_2^2 u'_1} = \overline{u_3^2 u'_1} = (\overline{u^2})^{3/2}\, h(r)$.

* In order to distinguish between correlations connecting two and three velocity components the correlations treated in the previous sections will be referred to in the following sections as "double correlations".

(Fig. 2 represents the double and triple correlation functions which do not vanish because of isotropy.)

It will be seen that the series development of the function $k(r)$ for small values of r starts with the term in r^3. For, developing u_1' in a series of powers of the variable $\xi_1 = x_1' - x_1 = r$, we obtain

$$\overline{u_1^2 u_1'} = \overline{u_1^3} + \overline{u_1^2 \frac{\partial u_1}{\partial \xi_1}} \xi_1 + \tfrac{1}{2} \overline{u_1^2 \frac{\partial^2 u_1}{\partial \xi_1^2}} \frac{\xi_1^2}{2} + \dots .$$

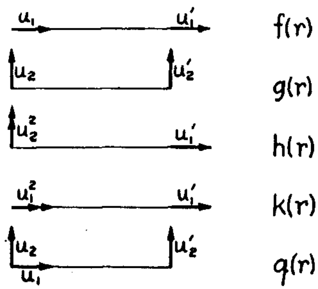


Fig. 2

Now, reflecting the x-axis all the coefficients of even powers of ξ_1 change their sign, and consequently these coefficients must vanish in the case of isotropic turbulence. Because of the homogeneity of turbulence

$$\overline{u_1^2 \frac{\partial u_1}{\partial \xi_1}} = \frac{1}{3} \frac{\partial \overline{u_1^3}}{\partial \xi_1},$$

also vanishes. Hence, the development of $\overline{u_1^2 u_1'}$ starts with the term

$$\tfrac{1}{6} \overline{u_1^2 \frac{\partial^3 u_1}{\partial \xi_1^3}} \xi_1^3$$

and correspondingly

$$k(r) = k'''(0) \frac{r^3}{6} + k^{\mathrm{v}}(0) \frac{r^5}{120} + \dots .$$

Until now it has been assumed that P and P' are situated on the x-axis. In order to obtain the general expression for $\overline{u_i u_j u_k'}$ we take $\overline{PP'}$ to be an arbitrary direction and denote by (p_1, p_2, p_3) and (p_1', p_2', p_3') the velocity components at P and P' along three mutually perpendicular lines whose direction cosines are (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3). In particular, we suppose that $l_1 = \xi_1/r$, $m_1 = \xi_2/r$, $n_1 = \xi_3/r$, so that the first line is in the direction $\overline{PP'}$.

We then obtain for the velocity components along the x, y, z axes:

$$\left.\begin{aligned} u_1 &= l_i p_i, & u_1' &= l_i p_i', \\ u_2 &= m_i p_i, & u_2' &= m_i p_i', \\ u_3 &= n_i p_i, & u_3' &= n_i p_i'. \end{aligned}\right\} \tag{34}$$

On the other hand, because of isotropy

$$\overline{p_1^2 p_1'} = (\overline{u^2})^{\frac{3}{2}} k(r),\ \overline{p_1 p_2 p_2'} = \overline{p_1 p_3 p_3'} = (\overline{u^2})^{\frac{3}{2}} q(r) \text{ and } \overline{p_2^2 p_1'} = \overline{p_3^2 p_1'} = (\overline{u^2})^{\frac{3}{2}} h(r),$$

all other combinations being equal to zero.

Using the expressions (34) and substituting for the mean values of the products of the p's the functions $k(r)$, $q(r)$ and $h(r)$ we find after some analysis that

$$\overline{u_i u_j u_k'} = (\overline{u^2})^{\frac{3}{2}} T_{ijk} = (\overline{u^2})^{\frac{3}{2}} \left\{ \xi_i \xi_j \xi_k \frac{k-h-2q}{r^3} + \delta_{ij} \xi_k \frac{h}{r} + \delta_{ik} \xi_j \frac{q}{r} + \delta_{jk} \xi_i \frac{q}{r} \right\}, \tag{35}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$).

It is seen that T_{ijk} is an odd function of the variables ξ_1, ξ_2, ξ_3. By interchanging P and P' we change ξ_1, ξ_2 and ξ_3 into $-\xi_1$, $-\xi_2$ and $-\xi_3$ respectively. Hence the mean values of triple products composed of one component measured at P and two components measured at P' can be expressed by the T_{ijk}'s. For instance

$$\overline{u_i u_j' u_k'} = -\overline{u_i' u_j u_k} = -\overline{u_k u_j u_i'} = -(\overline{u^2})^{\frac{3}{2}} T_{kji}.$$

This relation will be used in § 8.

We shall now show that because of the continuity relation between the velocity components the functions k and q can be expressed by h. Since u_i and u_j are independent of the variation of x_k', it follows from the continuity equation that

$$\frac{\partial}{\partial x_k'} (\overline{u_i u_j u_k'}) = 0. \tag{36}$$

Hence
$$\frac{\partial}{\partial \xi_k} (\overline{u_i u_j u_k'}) = 0 \quad \text{or} \quad \frac{\partial}{\partial \xi_k} T_{ijk} = 0. \tag{37}$$

By substituting (35) in (37) and carrying out the differentiations, we find

$$\xi_i \xi_j \left[\frac{k'-h'}{r^2} + \frac{2k-2h-6q}{r^3} \right] + \delta_{ij} \left(\frac{2h}{r} + \frac{2q}{r} + h' \right) = 0. \tag{38}$$

The symbols k' and h' denote differential quotients with respect to r. The k, h and q being only functions of r, the expressions in the brackets must vanish separately. This leads to the results

$$\left.\begin{aligned} k &= -2h, \\ q &= -h - \frac{r}{2}\frac{dh}{dr}. \end{aligned}\right\} \tag{39}$$

It was seen above that the development of $k(r)$ for small values of r starts with the term containing r^3. Because of (39) the same is true for $h(r)$ and $q(r)$.

The expression (35) and the relations (39) carry the general analysis of the triple correlations as far as we did in § 3 in the case of the double correlations.

7—The correlation between pressure and velocity

We shall now prove that the mean values $\overline{\varpi u_j'}$ ($j = 1$, 2 or 3), where ϖ is the pressure at (x_1, x_2, x_3), all vanish. Again, denote by (p_1', p_2', p_3') the velocity components at (x_1', x_2', x_3') along the lines whose direction cosines are given by (l_i, m_i, n_i) ($i = 1, 2$ or 3) and defined in the preceding paragraph. It is clear, by symmetry, that $\overline{\varpi p_j'}$ ($j = 2$ or 3) vanishes. It is, however, necessary to introduce the equation of continuity to show that $\overline{\varpi p_1'} = 0$. Then let us write

$$\varpi p_1' = \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}} s(r).$$

We find immediately from (34) that

$$\left.\begin{aligned} \overline{\varpi u_1'} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}[l_1 s(r)], \\ \overline{\varpi u_2'} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}[m_1 s(r)], \\ \overline{\varpi u_3'} &= \{\overline{\varpi^2}\}^{\frac{1}{2}}\{\overline{u^2}\}^{\frac{1}{2}}[n_1 s(r)]. \end{aligned}\right\} \tag{40}$$

Here again, the equation of continuity leads to the condition

$$\frac{\partial}{\partial \xi_i}(\overline{\varpi u_i'}) = 0$$

for all values of ξ_1, ξ_2, ξ_3. Transforming to spherical polar co-ordinates we see that

$$\frac{\partial}{\partial r}[r^2 \sin\theta\, s(r)] = 0$$

for all values of r and θ, so that

$$\frac{d}{dr}s(r) + \frac{2s(r)}{r} = 0.$$

Again, since s is regular at the origin, the appropriate solution is

$$s = 0.$$

Hence, $\overline{\varpi u'_j} = 0$ for $j = 1, 2$ or 3. Similarly, $\overline{\varpi' u_j} = 0$ for $j = 1, 2$ or 3.

B—Dynamics of isotropic turbulence

8—*The equation for the propagation of the correlation*

The equations of motion at the point $P(x_1, x_2, x_3)$ can be written

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu \nabla_x^2 u_i \tag{41}$$

for $i = 1, 2$ or 3, where $\nabla_x^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and ϖ denotes the pressure. Let us multiply equation (41) by the k-component u'_k of the velocity at $P'(x'_1, x'_2, x'_3)$, so that

$$u'_k \frac{\partial u_i}{\partial t} + u'_k u_j \frac{\partial u_i}{\partial x_j} = \frac{-u'_k}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu u'_k \nabla_x^2 u_i. \tag{42}$$

Consider now the expression $u'_k u_j \frac{\partial u_i}{\partial x_j}$. It will be seen that, in virtue of the equation of continuity,

$$u'_k u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i u_j u'_k). \tag{43}$$

Now $\overline{u_i u_j u'_k}$ is a function of ξ_1, ξ_2 and ξ_3 and obviously

$$\frac{\partial}{\partial x_j}(\overline{u_i u_j u'_k}) = -\frac{\partial}{\partial \xi_j}(\overline{u_i u_j u'_k}). \tag{44}$$

Furthermore,

$$\frac{-u'_k}{\rho}\frac{\partial \varpi}{\partial x_i} = \frac{1}{\rho}\frac{\partial}{\partial \xi_i}(\varpi u'_k), \tag{45}$$

and since $\overline{\varpi u'_k}$ vanishes everywhere, $\overline{\frac{-u'_k}{\rho}\frac{\partial \varpi}{\partial x_i}}$ also vanishes. Finally,

$$\nabla_x^2(\overline{u_i u'_k}) = \nabla^2(\overline{u_i u'_k}),$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2}.$$

Hence we obtain the equation

$$\overline{u'_k \frac{\partial u_i}{\partial t}} - \frac{\partial}{\partial \xi_j}(\overline{u_i u_j u'_k}) = \nu \nabla^2(\overline{u_i u'_k}). \tag{46}$$

In an analogous way we obtain

$$\overline{u_i \frac{\partial u_k'}{\partial t}} + \frac{\partial}{\partial \xi_j}(\overline{u_i u_j' u_k'}) = \nu \nabla^2 (\overline{u_i u_k'}). \tag{47}$$

We have shown in § 7 that

$$\overline{u_i u_j' u_k'} = -(\overline{u_i' u_j u_k}), \tag{48}$$

and substituting (48) in equation (47) we obtain

$$\overline{u_i \frac{\partial u_k'}{\partial t}} - \frac{\partial}{\partial \xi_j}(\overline{u_j u_k u_i'}) = \nu \nabla^2 (\overline{u_i u_k'}). \tag{49}$$

Adding the equations (46) and (49) and remembering that $\overline{u_i u_k'} = \overline{u^2} R_{ik}$ and $\overline{u_i u_j u_k'} = (\overline{u^2})^{3/2} T_{ijk}$, we obtain finally

$$\frac{\partial}{\partial t}(\overline{u^2} R_{ik}) - (\overline{u^2})^{3/2} \frac{\partial}{\partial \xi_j}(T_{ijk} + T_{kji}) = 2\nu \overline{u^2} \nabla^2 R_{ik}. \tag{50}$$

In view of the form given for the tensors R_{ik} and T_{ijk} in equations (3) and (35) respectively, equation (50) may clearly be reduced to a differential equation connecting the functions f, g, k, q, h. By using the equation of continuity in the form given by equations (7) and (39) we can eliminate g, k and q and obtain a partial differential equation connecting f and h. The analysis can be carried out by choosing any component of R_{ik}. Then equation (50) contains terms multiplied by $\xi_i \xi_k$ and terms which are functions of r alone. Equating the terms containing $\xi_i \xi_k$ we obtain a relation for $\partial/\partial t\,(f-g)$ and equating the terms which are functions of r a relation for $\partial g/\partial t$. Eliminating $\partial g/\partial t$ the following equation for f results

$$\frac{\partial(f\overline{u^2})}{\partial t} + 2(\overline{u^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r} h\right) = 2\nu \overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{51}$$

We call (51) the fundamental equation for the propagation of the correlation function $f(r)$.

9—*The decay of turbulence*

Before proceeding with the discussion of the solutions of equation (51) we shall show how Taylor's equation for the decay of energy and the equation for the decay of vorticity already given by the senior author (Kármán 1937*a*) can be obtained as deductions from equation (51).

$$\text{For } r = 0, f = 1 \text{ and } \frac{dh}{dr} + \frac{4}{r} h = 0 \quad (\text{cf. § 6}).$$

Hence we obtain from (51)

$$\frac{\partial \overline{u^2}}{\partial t} = 2\nu \overline{u^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}\right)_{r=0}, \tag{52}$$

or with

$$f = 1+\tfrac{1}{2}f_0'' r^2+\tfrac{1}{24}f_0^{\mathrm{iv}} r^4+\ldots, \tag{53}$$

$$\frac{d\overline{u^2}}{dt} = 10\nu f_0''\overline{u^2}. \tag{54}$$

Now Taylor's definition of the length λ is given by the equation

$$\lambda^2 = -2/g_0''. \tag{54a}$$

We have shown (equation 10) that $f'' = \tfrac{1}{2}g_0''$. Hence

$$\frac{d\overline{u^2}}{dt} = -10\nu\frac{\overline{u^2}}{\lambda^2}. \tag{55}$$

Or if, considering Taylor's problem of the decay of turbulence behind a honeycomb, we substitute $\dfrac{d}{dt} = \dfrac{1}{U}\dfrac{d}{dx}$,

$$\frac{1}{U}\frac{d}{dx}\overline{u^2} = -10\nu\frac{\overline{u^2}}{\lambda^2}, \tag{56}$$

which is Taylor's equation for the decrease of the mean kinetic energy of turbulence.

Now we shall substitute the expansions given by equation (53) in equation (51) and equate the coefficients of the terms containing r^2. Concerning the triple correlation function $h(r)$ we notice that

$$h(r) = \tfrac{1}{6}h_0''' r^3+\text{higher terms}. \tag{57}$$

Hence we find

$$\frac{d}{dt}[\tfrac{1}{2}f_0''\overline{u^2}]+\tfrac{7}{3}h_0'''(\overline{u^2})^{\frac{3}{2}} = \tfrac{7}{3}\nu f_0^{\mathrm{iv}}\overline{u^2}. \tag{58}$$

Obviously $f_0''\overline{u^2} = -\overline{u^2}/\lambda^2$. On the other hand, we find the mean square of the vorticity components using the formulae (19), (20) and (21)

$$\overline{\omega_1^2} = \overline{\omega_2^2} = \overline{\omega_3^2} = 5\overline{u^2}/\lambda.$$

Hence equation (58) can be interpreted as the equation for the decay of vorticity. Let us denote $\overline{\omega_1^2}+\overline{\omega_2^2}+\overline{\omega_3^2}$ by $\overline{\omega^2}$ then $\overline{\omega^2} = 15\overline{u^2}/\lambda^2$ and from (58) follows

$$\frac{d\overline{\omega^2}}{dt}-70h_0'''(\overline{u^2})^{\frac{3}{2}} = -\frac{14}{3}\nu\overline{\omega^2}\lambda^2 f_0^{\mathrm{iv}}.$$

$\lambda^2 f_0^{\text{iv}}$ is the reciprocal value of the square of a length; in order to obtain conformity with the energy equation (55) we put $\frac{1}{\lambda_\omega^2} = \frac{7}{15}\lambda^2 f_0^{\text{iv}}$. Then

$$\frac{d\overline{\omega^2}}{dt} - 70 h_0'''(\overline{u^2})^{\frac{3}{2}} = -10\nu\frac{\overline{\omega^2}}{\lambda_\omega^2}. \tag{59}$$

From the equations of motion the senior author obtained the equation* (Kármán 1937*a*)

$$\frac{d\overline{\omega^2}}{dt} - 2\overline{\omega_i\,\omega_k\frac{\partial u_i}{\partial x_k}} = -10\nu\frac{\overline{\omega^2}}{\lambda_\omega^2}. \tag{60}$$

Equations (59) and (60) are identical. Because of the continuity equation

$$-2\,\overline{\omega_i\,\omega_k\frac{\partial u_i}{\partial x_k}} = \overline{\left(\frac{\partial^2(u_j u_i)}{\partial x_j\partial x_k} - \frac{\partial^2(u_j u_k)}{\partial x_j\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)}.$$

Now the mean values, which constitute the expression on the right side, can be calculated by differentiation of the equation (35). For instance,

$$\overline{\frac{\partial^2(u_i u_j)}{\partial x_j\partial x_k}\frac{\partial u_i}{\partial x_k}} = \left[\frac{\partial^3(\overline{u_i u_j u_i'})}{\partial\xi_j\partial\xi_k\partial\xi_k}\right]_{\xi_i=\xi_j=\xi_k=0},$$

and so on.

Carrying out the computations the identity of equations (59) and (60) can be shown without difficulty.

It is seen that the equation for the decay of turbulence and for the decay of vorticity follow from the general equation for the spread of correlations by development of the correlation functions in powers of the distance r. The expression for the decay of vorticity contains two terms: one corresponds to the change of vorticity by deformation of the vortex tubes, the other corresponds to the action of viscosity.

* As the senior author found the equation (60) he realized that the condition of isotropy alone does not lead to a further reduction of the equation. However, he thought that in a random isotropic turbulence the expression containing the triple correlations should vanish because the extension and the contraction of portions of the vortex tubes should be equally probable in an ideal fluid. As the authors carried out their first general analysis of correlations, it seemed that a general proof could be found for the vanishing of the triple correlations. Correspondingly, the senior author stated in his papers on the subject (1937 *a*, *b*) that the triple correlations and the term in question in equation (60) vanish because of isotropy. Closer analysis showed that this statement is erroneous. To make the triple correlations vanish, some further physical assumption on the random character of turbulence is necessary.

10—*Self-preserving correlation functions*

We write equation (51) in the form

$$\frac{\partial(f\overline{u^2})}{\partial t} = 2\nu\overline{u^2}\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)\left[\frac{\partial f}{\partial r}-\frac{\sqrt{\overline{u^2}}}{\nu}h\right]. \tag{61}$$

Obviously, the influence of the triple correlations depend on the relative magnitude of the quantities $\frac{\partial f}{\partial r}$ and $\frac{\sqrt{\overline{u^2}}}{\nu}h$. Let us consider first the case of "small Reynolds number" of the turbulence, i.e. let us assume that $\frac{\partial f}{\partial r} \gg \frac{\sqrt{\overline{u^2}}h}{\nu}$. Then neglecting the triple correlation function h, we write

$$\frac{\partial(f\overline{u^2})}{\partial t} = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}\right), \tag{62}$$

or eliminating $\overline{u^2}$ by use of equation (54)

$$\frac{\partial f}{\partial t} = 2\nu\left(\frac{\partial^2 f}{\partial r^2}+\frac{4}{r}\frac{\partial f}{\partial r}-5f''_0 f\right). \tag{63}$$

We notice that in this case $f(r,t)$ is determined by the values of $f(r,t_0)$, i.e. if the correlation function f is given for $t=t_0$ it is given for any arbitrary time $t>t_0$.

We shall consider a particular set of solutions of equation (63). We notice that this equation reduces to an ordinary differential equation if we suppose f is a function of $\chi = \frac{r}{\sqrt{(\nu t)}}$ only. When this substitution is made, we obtain the equation

$$f''+\left(\frac{4}{\chi}+\frac{\chi}{4}\right)f'-5f''(0)f = 0, \tag{64}$$

where dashes denote differentiations with regard to χ; we shall speak of the correlation functions given by the solution of equation (64) as "self-preserving", since the form of these curves is the same at all instants although the actual length scale varies. We shall now denote by α the arbitrary constant $-f''(0)$.

First of all it may be pointed out that equation (64) is related to the confluent hypergeometric equation. In fact, the solution we require (i.e. the one which satisfies $f = 1$ and $f' = 0$ when $\chi = 0$) is equal to

$$f(\chi) = 2^{15/4}\chi^{-5/2}e^{-\chi^2/16}M_{(10\alpha-\frac{5}{4}),\frac{3}{4}}\left(\frac{\chi^2}{8}\right), \tag{65}$$

where $M_{k,m}(z)$ is the same solution of the confluent hypergeometric equation as that defined by Whittaker and Watson (1927, p. 337, para. 16.1) and denoted by this symbol.

When $\alpha < \frac{1}{4}$ it will be seen, after some reduction, that (Whittaker and Watson 1927, p. 352, example 1)

$$f(\chi) = \frac{\Gamma(5/2)}{\Gamma(10\alpha)\,\Gamma(\frac{5}{2} - 10\alpha)} \int_0^1 \tau^{(10\alpha - 1)} (1 - \tau)^{(\frac{3}{2} - 10\alpha)} e^{-\frac{\chi^2}{8}\tau} d\tau. \tag{66}$$

When $\alpha = \frac{1}{4}$ we obtain the solution

$$f(\chi) = e^{-\chi^2/8}. \tag{67}$$

When $\alpha > \frac{1}{4}$ the solution is given by the integral of the same integrand round a particular contour (Whittaker and Watson, 1927, p. 257).

Let us now consider the decay of turbulence when the correlation function is one of these special types. Returning to equation (54) we see that

$$\frac{\partial \overline{u^2}}{\partial t} = -10 \frac{\overline{u^2}}{t} \alpha, \tag{68}$$

and integrating we find

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}} \left(\frac{t}{t_0}\right)^{5\alpha}, \tag{69}$$

where the condition $\overline{u^2} = \overline{u_0^2}$ when $t = t_0$ has been applied.

Assuming, for the moment, that the turbulence produced by a particular grid or honeycomb is of the type which has a self-preserving correlation function and starting from some point outside the wind shadow, where the fluid passes at the time t_0, we write $t = t_0 + x/U$ (U denotes the velocity of the main flow and x is the distance measured downstream). Then we may write equation (69) in the form

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}} \left(1 + \frac{x}{Ut_0}\right)^{5\alpha}, \tag{70}$$

where $\overline{u^2} = \overline{u_0^2}$, when $x = 0$.

The quantity Ut_0 is a length which we can relate to the value λ_0 of the length λ (defined in equation (54a) above) at the origin from which x is measured. For

$$\left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} = \frac{1}{\nu t} f''(0) = -\frac{\alpha}{\nu t},$$

so that, with $\left(\frac{\partial^2 f}{\partial r^2}\right)_{r=0} = -1/\lambda^2$

$$\lambda^2 = \frac{\nu t}{\alpha}, \tag{71}$$

and therefore $$t_0 = \frac{\alpha\lambda_0^2}{\nu}.$$

Finally, from equation (70),

$$\frac{U}{\sqrt{\overline{u^2}}} = \frac{U}{\sqrt{\overline{u_0^2}}}\left(1+\frac{x\nu}{\alpha\lambda_0^2 U}\right)^{5\alpha}, \tag{72}$$

where $\overline{u^2} = \overline{u_0^2}$, $\lambda = \lambda_0$ when $x = 0$.

The results given by equations (71) and (72) would be formally equivalent with Taylor's results, when $\alpha = \frac{1}{5}$. However, the corresponding correlation function f which is given in fig. 3 as calculated by numerical integration from the integral obtained by integrating (66) by parts is very

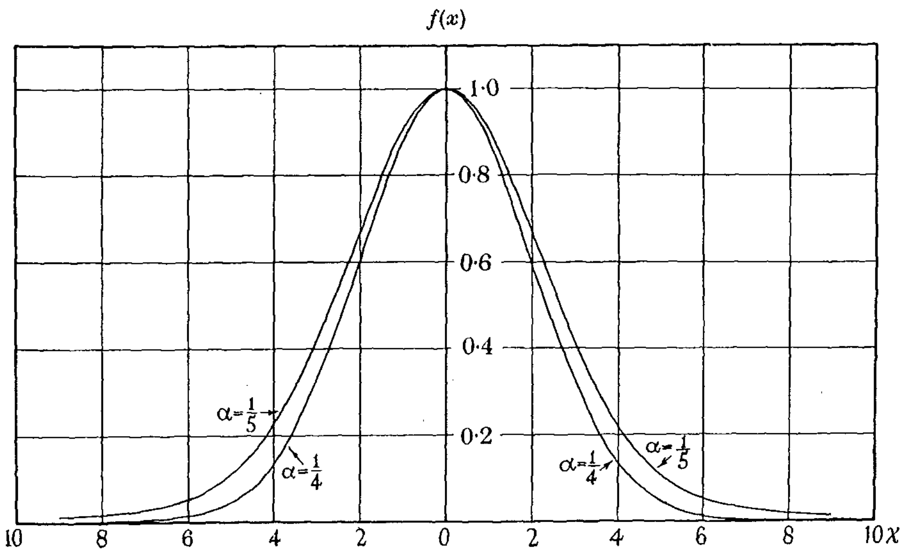


FIG. 3

different from those measured by Simmons in such cases, in which Taylor's linear law of the decay of turbulence apparently holds. Hence the coincidence between these equations and Taylor's result is rather formal. We come back to this question in the last section of this paper. The correlation function f corresponding to $\alpha = \frac{1}{4}$ (see equation (67)) is also included in fig. 3.

11—*Possible solution for large Reynolds numbers**

It was seen that if the triple correlations are neglected the double correlation function $f(r)$ is determined for all times $t > t_0$ provided $f(r)$ is known for $t = t_0$. In the general case $\partial f/\partial t$ depends also on the triple correlation function $h(r)$. Now a further equation can be obtained from the equations of motion for $\partial h/\partial t$; however, this new equation contains terms with the quadruple correlations and so on. Hence, for an arbitrary value of the Reynolds number the problem is too complicated for analytical treatment.

However, certain interesting results can be obtained for the limiting case of very large Reynolds numbers under the assumption that the correlation functions $f(r, t)$ and $h(r, t)$ are independent of viscosity with the exception of small values of the distance r. Let us consider the problem of turbulence created by passing a uniform windstream through a grid or similar turbulence-producing device and let us assume geometrically similar arrangements. Then, denoting the mean velocity of the windstream by U and a characteristic linear dimension of the device mentioned by M, it seems evident by dimensional considerations that f has the form

$$f\left(\frac{r}{\sqrt{(\nu t)}}, \frac{r}{M}, \frac{Ut}{M}\right).$$

If the correlation functions are—according to the above assumption—independent of viscosity, this means that $f(r, t)$ does not depend on the variable $\chi = \dfrac{r}{\sqrt{(\nu t)}}$ and is only a function of the variables r/M and Ut/M. Let us now assume in addition that the correlation functions preserve their shape and only their scale is changing. Obviously this assumption amounts to the statement that both $f(r, t)$ and $h(r, t)$ are functions of one variable $\psi = r/L$ only, where L is a function of M and Ut.

It must be noticed that both assumptions—namely that the correlation functions are independent of χ and that they preserve their shape—shall be made only for large values of $\chi = \dfrac{r}{\sqrt{(\nu t)}}$.

Then introducing $f(\psi)$ and $h(\psi)$ in equation (51), we obtain the following expression:

$$-\frac{df}{d\psi}\psi\frac{\overline{u^2}}{L}\frac{dL}{dt}+f\frac{d\overline{u^2}}{dt}+2\frac{(\overline{u^2})^{\frac{3}{2}}}{L}\left(\frac{dh}{d\psi}+\frac{4h}{\psi}\right) = 2\nu\frac{\overline{u^2}}{L^2}\left(\frac{d^2f}{d\psi^2}+\frac{4}{\psi}\frac{df}{d\psi}\right). \quad (73)$$

* This section was inserted by the senior author in Sept. 1937, especially after reading G. I. Taylor's contribution (1937).

Let us consider the quantity $\frac{\sqrt{\overline{u^2}}L}{\nu}$ as the Reynolds number related to the problem, then for large values of this number it appears justified to neglect the term on the right side. Furthermore, according to (55)

$$\frac{d\overline{u^2}}{dt} = -10\nu\frac{\overline{u^2}}{\lambda^2}.$$

Hence, from (73) it follows that

$$-\frac{df}{d\psi}\psi\frac{\overline{u^2}}{L}\frac{dL}{dt} - f\frac{10\nu\overline{u^2}}{\lambda^2} + \left(\frac{dh}{d\psi} + \frac{4h}{\psi}\right)2\frac{(\overline{u^2})^{\frac{3}{2}}}{L} = 0. \tag{74}$$

Obviously equation (74) can be satisfied only when the coefficients which are functions of t without being functions of ψ, are proportional, i.e. their ratios are constants. Hence

$$\frac{\sqrt{\overline{u^2}}\lambda^2}{L\nu} = A, \tag{75}$$

$$\frac{dL}{dt} = B\sqrt{\overline{u^2}}, \tag{76}$$

where A and B are numerical constants.

We easily obtain from (75) and (76) a differential equation for $L(t)$. We substitute in (75) from (55) $\frac{\lambda^2}{\nu} = -10\overline{u^2}\Big/\frac{d}{dt}\overline{u^2}$ and eliminate $\overline{u^2}$ from (75) and (76). Thus we obtain

$$L\frac{d^2L}{dt^2} = -\frac{5}{AB}\left(\frac{dL}{dt}\right)^2. \tag{77}$$

The general solution of (77) is

$$L = L_0\left(1 + \frac{t}{t_0}\right)^{\frac{5}{5+AB}}, \tag{78}$$

where L_0 and t_0 are arbitrary constants. The origin of the time t being arbitrary, we may write

$$L = L_0\left(\frac{t}{t_0}\right)^{\frac{5}{5+AB}}. \tag{79}$$

Introducing the expression (79) in (76) we obtain by differentiation

$$\sqrt{\overline{u^2}} = \frac{5}{5+AB}L_0\left(\frac{t}{t_0}\right)^{\frac{5}{5+AB}}\frac{1}{t}, \tag{80}$$

or denoting the value of $\sqrt{\overline{u^2}}$ at $t = t_0$ by $\sqrt{\overline{u_0^2}}$

$$\sqrt{\overline{u^2}} = \sqrt{\overline{u_0^2}}\left(\frac{t}{t_0}\right)^{\frac{-AB}{5+AB}}. \tag{81}$$

Finally from (55)

$$\lambda^2 = \left(5+\frac{25}{AB}\right)\nu t. \tag{82}$$

It is seen that equations (81) and (82) with $\frac{1}{\alpha} = 5+\frac{25}{AB}$ are identical with the corresponding equations (69) and (71) obtained in § 10 for the case of the small Reynolds number.

There is one case which is not included in equation (77), namely when $L =$ const. say $L = L_0$. Then from (75) with $\frac{\lambda^2}{\nu} = -5\sqrt{\overline{u^2}}\Big/\frac{d}{dt}\sqrt{\overline{u^2}}$ it follows that

$$\frac{1}{\sqrt{\overline{u^2}}} = -\frac{5}{AL_0}t \tag{83}$$

and

$$\lambda^2 = 5\nu t. \tag{84}$$

Let us compare the results obtained with the researches of Taylor and Dryden. Our last equations (83) and (84) are identical with Taylor's results; especially if we assume with Taylor that L_0 is proportional to M, which is in this case evident from dimensional consideration. However, Taylor does not have the solutions (81) and (82). The reason is that Taylor found the equation (75) with a remarkable vision for the relations between the quantities involved; however, instead of (76) he assumed L proportional to M, i.e. a fixed ratio between the scale of turbulence and the linear size of the turbulence-producing device.

In both cases—using Taylor's assumption or our broader relations (75) and (76)—the theory leads to the conclusion that the scale and distribution of turbulence in a windstream is independent of the speed of the windstream and so is the ratio $U/\sqrt{\overline{u^2}}$ measured at a certain distance x from the grid. The difference is that according to our theory the scale of turbulence may increase downstream, while according to Taylor's assumption, it remains unchanged. Instead of $L = \text{const.} \times M$, we obtain in general $L = M \times$ function of Ut/M, and in the case of "self-preserving" turbulence the function involved is proportional to a certain power of Ut/M. It appears that further experimental results will decide whether Taylor's assumptions are not too narrow and whether the equations (79) and (80) correspond more closely to the experimental facts.

It is interesting that as far as the laws for the decay of turbulence and for the spread of correlation curves are concerned, our analysis of correlations leads essentially to the same results as the recently published theory of Dryden's (1937) which is based on entirely different conceptions. It appears that the next step in the development of the theory should be to find the physical mechanism which is behind the mathematical relations (75) and (76), especially the mechanism which tends to increase the scale of turbulence without the action of the viscosity.

References

Dryden, H. L. 1937 *J. Aero. Sci.* **4**, 273.
Kármán, T. de 1937 *a* *Proc. Nat. Acad. Sci., Wash.*, **23**, 98.
— 1937 *b* *J. Aero. Sci.* **4**, 131.
Lamb 1932 "Hydrodynamics", 6th ed. pp. 208–9.
Taylor, G. I. 1935 *Proc. Roy. Soc.* A, **151**, 421–78.
— 1937 *J. Aero. Sci.* **4**, 311.
Whittaker, E. T. and Watson 1927 "Modern Analysis", 4th ed. p. 337.

The spectrum of turbulence

By G. I. Taylor, F.R.S.

(*Received* 1 *December* 1937)

When a prism is set up in the path of a beam of white light it analyses the time variation of electric intensity at a point into its harmonic components and separates them into a spectrum. Since the velocity of light for all wave-lengths is the same, the time variation analysis is exactly equivalent to a harmonic analysis of the space variation of electric intensity along the beam. In a recent paper Mr Simmons (Simmons and Salter 1938) has shown how the time variation in velocity at a field point in a turbulent air stream can be analysed into a spectrum. In the present paper it is proposed to discuss the connexion between the spectrum of turbulence, measured at a fixed point, and the correlation between *simultaneous* values of velocity measured at two points.

If u, the component at a fixed point of turbulent motion in the direction of the main stream in a wind tunnel, is resolved into harmonic components the mean value of u^2 may be regarded as being the sum of contributions from all frequencies. If $\overline{u^2}\,F(n)\,dn$ is the contribution from frequencies between n and $n+dn$, then

$$\int_0^\infty F(n)\,dn = 1. \tag{1}$$

If $F(n)$ is plotted against n, the diagram so produced is a form of the spectrum curve.

The proof that $\overline{u^2}$ may be regarded as being the sum of contributions from all the harmonic components has been given by Rayleigh, using a form of Parseval's theorem. If $u = \phi(t)$ and

$$\left.\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(t)\cos\kappa t\,dt, \\ I_2 &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(t)\sin\kappa t\,dt. \end{aligned}\right\} \tag{2}$$

Rayleigh showed that

$$\int_{-\infty}^{+\infty} [\phi(t)]^2\,dt = \pi\int_0^{\infty} (I_1^2 + I_2^2)\,d\kappa, \tag{3}$$

or if $\kappa = 2\pi n$, so that n represents the number of cycles per second,

$$\int_{-\infty}^{+\infty} [\phi(t)]^2\,dt = 2\pi^2\int_0^{\infty} (I_1^2 + I_2^2)\,dn. \tag{4}$$

If the integrals on the right-hand side of (2) and the left-hand side of (3) are taken over a long time T instead of infinity the left-hand side of (3) is $T\overline{u^2}$, so that

$$\overline{u^2} = 2\pi^2\int_0^{\infty} \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2 + I_2^2}{T}\right) dn. \tag{5}$$

The quantity $2\pi^2 \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2 + I_2^2}{T}\right)$ is therefore the contribution to $\overline{u^2}$ which arises from the components of frequency between n and $n + dn$, i.e.

$$2\pi^2 \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2 + I_2^2}{T}\right) = F(n). \tag{6}$$

Connexion between spectrum curve and correlation curve

Now consider two cases: (*a*) where the variation in u is due to eddies of small extent which are carried by a wind stream of velocity U past the fixed point; (*b*) the variation is due to large eddies carried in the wind stream. In case (*a*) the fluctuations at the fixed point will be much more rapid than they are in case (*b*). The spectrum analysis in case (*b*) will therefore show greater values of $F(n)$ for small values of n than in case (*a*).

It is clear when the eddies are large the correlation R_x between simultaneous values of u at distance x apart must fall away with increasing x more slowly than when the eddies are small. One may therefore anticipate that when the (R_x, x) curve has a small spread in the x co-ordinate the $F(n)$ curve will extend to large values of n and vice versa.

If the velocity of the air stream which carries the eddies is very much greater than the turbulent velocity, one may assume that the sequence of changes in u at the fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point, i.e. one may assume that

$$u = \phi(t) = \phi\left(\frac{x}{U}\right), \tag{7}$$

where x is measured upstream at time $t = 0$ from the fixed point where u is measured. In the limit when $u/U \to 0$ (7) is certainly true. Assuming that (7) is still true when u/U is small but not zero, R_x is defined as

$$R_x = \frac{\overline{\phi(t)\,\phi\left(t+\frac{x}{U}\right)}}{\overline{u^2}} \tag{8}$$

We now introduce another expression analogous to (3). It can be shown that*

$$\int_{-\infty}^{+\infty} \phi(t)\,\phi\left(t+\frac{x}{U}\right) dt = 2\pi^2 \int_0^\infty (I_1^2 + I_2^2) \cos\frac{2\pi n x}{U}\, du, \tag{9}$$

where I_1 and I_2 have the same meaning as in (3).

Substituting for $I_1^2 + I_2^2$ from (6), (9) becomes

$$\frac{\overline{\phi(t)\,\phi\left(t+\frac{x}{U}\right)}}{\overline{u^2}} = \int_0^\infty F(n) \cos\frac{2\pi n x}{U}\, dn; \tag{10}$$

hence from (8)

$$R_x = \int_0^\infty F(n) \cos\frac{2\pi n x}{U}\, dn. \tag{11}$$

It will be noticed that the form of (11) is very similar to that of the Fourier

* This formula can be deduced from the theorem 9·09 given on p. 70 of Norbert Wiener's *The Fourier Integral*, Camb. Univ. Press, 1933.

integral. The Fourier integral theorem is usually expressed by the pair of formulae

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\mu) \cos \mu x \, d\mu, \tag{12}$$

$$g(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos \mu x \, dx. \tag{13}$$

When $f(x)$ is symmetrical, so that $f(x) = f(-x)$, (12) and (13) may be written

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\mu) \cos \mu x \, d\mu, \tag{14}$$

$$g(\mu) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \mu x \, dx. \qquad 15)$$

Comparing (11) and (14) it will be seen that if

$$\mu = \frac{2\pi n}{U}, \quad f(x) = R_x, \quad g(\mu) = \frac{UF(n)}{2\sqrt{2\pi}},$$

then (11) and (14) are identical.

Making these substitutions in (15), the following expression is found for $F(n)$:

$$F(n) = \frac{4}{U} \int_{0}^{\infty} R_x \cos \frac{2\pi n x}{U} \, dx. \tag{16}$$

It seems therefore that R_x and $\dfrac{UF(n)}{2\sqrt{2\pi}}$ are Fourier transforms of one another.

If $F(n)$ is observed we can calculate R_x using (11), and if R_x is observed we can calculate the spectrum curve $F(n)$ using (16).

Comparison with observation

Measurements have been made by Mr L. F. G. Simmons of R_x and of $F(n)$ at a point 6 ft. 10 in. from a turbulence-producing grid with a mesh 3×3 in. at wind speeds $U = 15, 20, 25, 30$ and 35 ft./sec. It was found that except very close to $x = 0$, R_x is nearly independent of U within that range.* When R_x is independent of U it will be seen from (16) that $UF(n)$ must be a function of n/U.

Accordingly Mr Simmons' measurements of $F(n)$ for all values of U have

* A similar result has been obtained by Dryden, N.A.C.A. report 581, 1937.

been plotted on the same diagram (figs. 1 and 1 *a*), in which the ordinates are $UF(n)$ and the abscissae are n/U. The fact that the points fall so closely on one curve is very satisfactory evidence that the measurements of $F(n)$ are accurate.

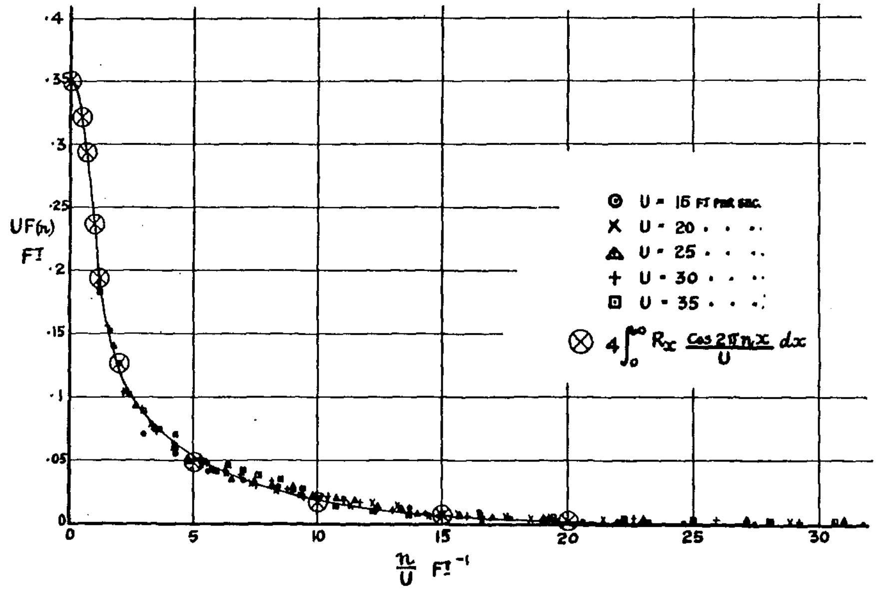


FIG. 1

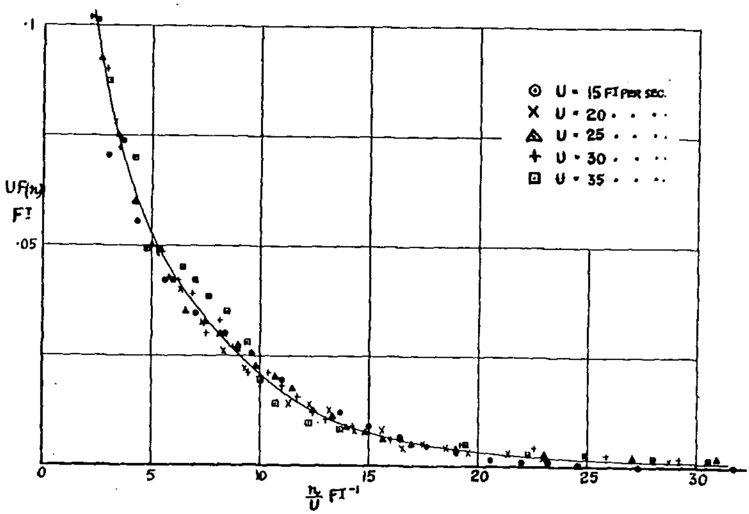


FIG. 1 *a*

Mr Simmons' measurements of R_x are shown in fig. 2. The values of $F(n)$ calculated using the measured values of R_x in (16) are shown in fig. 1, but as the points corresponding with the lower part of fig. 1 are rather close together an enlarged version is shown in fig. 1*a*. The values of R_x calculated using the measured values of $F(n)$ in (11) are shown in fig. 2.

It will be seen that the agreement in both cases is good.

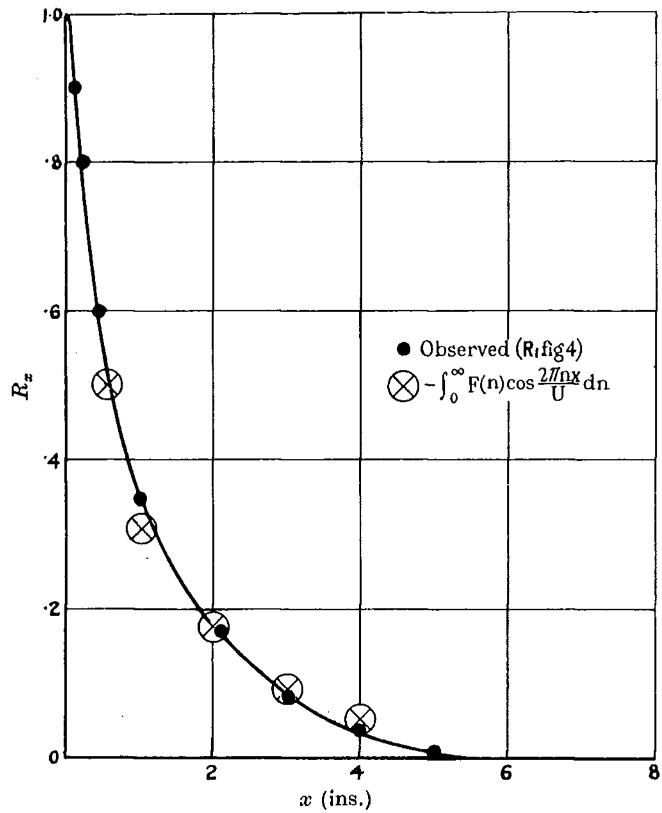


FIG. 2

It has been seen that the points in fig. 1 seem to fall on one curve, as is predicted by (16), when it is assumed that R_x is independent of U. On the other hand it is known that the curvature of the R_x curve at its vertex is not independent of U. This curvature is defined (Taylor 1935) by means of a length λ, where

$$\frac{1}{\lambda^2} = 2 \underset{x\to 0}{\mathrm{Lt}} \left(\frac{1-R_x}{x^2}\right). \tag{17}$$

If the turbulence is isotropic experiments show that λ is proportional to $U^{-\frac{1}{2}}$

To find what effect this variation in λ with U may be expected to produce in the $F(n)$ curve, λ may be expressed in terms of $F(n)$. When n is small $\cos\frac{2\pi nx}{U}$ may be replaced in (11) by $1-\frac{2\pi^2x^2n^2}{U^2}$. Hence

$$\frac{1}{\lambda^2}=\frac{4\pi^2}{U^2}\int_0^\infty n^2F(n)\,dn. \quad (18)$$

Since (18) can be written in the form

$$\frac{1}{\lambda^2}=4\pi^2\int_0^\infty \frac{n^2}{U^2}\,UF(n)\frac{dn}{U}, \quad (19)$$

λ must be independent of U if the $\{UF(n), n/U\}$ curve is independent of U. This deduction is inconsistent with the observed fact that λ is proportional to $U^{-\frac{1}{2}}$.

The explanation of this apparent discrepancy is that the value of $\int_0^\infty n^2F(n)\,dn$ depends chiefly on the values of $F(n)$ for large values of n, i.e. on the parts of figs. 1 and 1 *a* where the points are so close to the axis that variations in their height above it are hardly visible. In fig. 3 the vertical scale of $UF(n)$ has been enlarged very greatly. It will be seen that above $n/U = 16$ the $UF(n)$ curves separate, that for $U = 15$ ft./sec. falling below those for 20 and 35 ft./sec.

Calculation of λ from the spectrum curve

To determine λ from the spectrum curve the integral (19) must be evaluated using the values of $UF(n)$ taken from figs. 1 and 3. It is instructive to tabulate the contributions to this integral which arise from various ranges of n/U. These are set forth in Table I, where they are expressed in ft.-sec. units. It will be seen that when $U = 35$ about half of the integral is due to components for which $n/U > 30$, in spite of the fact that the highest

Table I. Contributions to $\int_0^\infty \frac{n^2}{U^2}F(n)\,dn$ expressed in ft.-sec. units

n/U	$U = 15$	$U = 20$	$U = 35$
0–16	24·0	24·0	24·0
17–30	7·7	19·0	23·9
31–∞	0	15·5	44·4
Total	31·7	58·5	92·3

value of $F(n)$ in this range is only $\frac{1}{200}$ of its maximum value (namely 0·35 when $n=0$). The value of λ in feet is found by inserting the numbers given at the foot of Table I for the integral in (19). Thus when $U = 35$ ft./sec. $\lambda = (4\pi^2 \times 92{\cdot}3)^{-\frac{1}{2}} = 0{\cdot}00165$ ft. $= 0{\cdot}50$ cm. The values of λ so calculated are given in column 2, Table II.

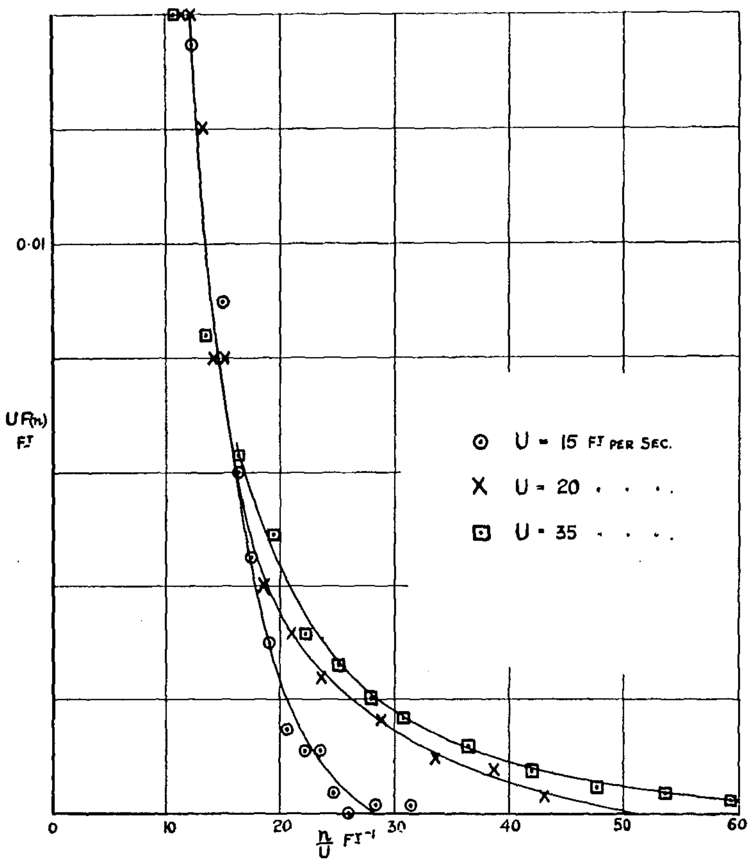


FIG. 3

VALUE OF $F(n)$ AT $n = 0$

Putting $x = 0$ in (16)

$$U[F(n)]_{n=0} = 4\int_0^\infty R_x dx.$$

By numerical integration of the measured R_x curve in fig. 2 it is found that

$$\int_0^\infty R_x dx = 1{\cdot}07\,\text{in.} = 0{\cdot}089\,\text{ft.},$$

so that, when $n = 0$,

$$U[F(n)]_{n=0} = 4 \times 0{\cdot}089 = 0{\cdot}35\,\text{ft.}$$

This upper limit is marked in fig. 1.

Proof that turbulence is isotropic

Though the theory and measurements so far discussed do not involve any assumption as to whether or not the turbulence is statistically isotropic, yet since the theory of isotropic turbulence has been discussed so completely it is worth while to describe measurements which prove that the turbulence was in fact isotropic.

It has been shown by Kármán (1937) that when turbulence is isotropic there is a definite relationship between the correlation curves R_x and R_y. Kármán defines two correlation functions R_1 and R_2. R_1 is the correlation between components of velocity along the line AB, where A and B are the points at which the velocities are measured, R_2 is the correlation between components at right angles to AB. In isotropic turbulence R_1 and R_2 are functions of r only where r is the length AB. When correlation measurements are made in a wind tunnel by means of a hot wire, only the component parallel to the length of the tunnel produces any appreciable effect on the hot wire. If therefore A and B are situated on a line parallel to the mean wind stream the correlation R_x is identical with Kármán's R_1. If A and B are situated on a line perpendicular to the stream the correlation R_y is identical with Kármán's R_2.

Kármán's relationship between R_1 and R_2, namely

$$R_2 = R_1 + \tfrac{1}{2} r \frac{dR_1}{dr}, \tag{20}$$

is therefore a relationship between the correlations R_x and R_y which have been measured. The measured values of R_x or R_1 are given in fig. 2, and they are repeated in fig. 4. To this curve the (negative) values of $\tfrac{1}{2}r(dR_1/dr)$ are added and the calculated values of R_y or R_2 thus obtained are shown in fig. 4. The values of R_y measured by Mr Simmons at 6 ft. 10 in. behind a 3×3 in. grid are also shown in fig. 4. It will be seen that Kármán's relationship (20) is very well verified, and it may fairly be concluded that the turbulence at 6 ft. 10 in. behind a 3 × 3 in. grid in a wind tunnel is isotropic.

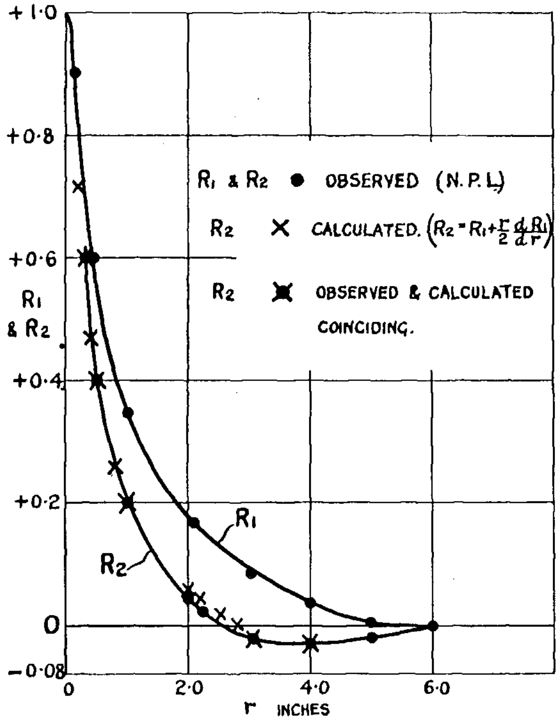


FIG. 4

CALCULATION OF λ FROM MEASURED RATE OF DISSIPATION OF ENERGY

In the case of isotropic turbulence λ can be found by measuring the rate of dissipation of energy. This can be found by measuring the rate of decay of the mean kinetic energy of turbulent motion.

Fig. 5 shows the measured values of U/u', $(u' = \sqrt{\overline{u^2}})$, in the air stream in which $F(n)$ and R_x were measured. It will be seen that in this case U/u' increases linearly* with x, the distance down stream from the grid. I have shown (Taylor 1935) that U/u' increases linearly with x when λ satisfies the following relationship:

$$\frac{\lambda}{M} = A\sqrt{\frac{\nu}{u'M}}, \tag{21}$$

* This is not a general law. Cases where U/u' does not increase linearly have been observed.

where M is the mesh size of the grid producing the turbulence. Conversely, when U/u' increases linearly with x, λ must be related to u' by the equation (21). By measuring the slope of the line which passes through the observed points in fig. 5, I find that A in (21) is 2·12. At the point where the spectrum measurements were made, 6 ft. 10 in. from the grid, $U/u' = 33{\cdot}5$. Since $M = 3$ in. $= 7{\cdot}62$ cm, and $\nu = 0{\cdot}148$, (21) becomes

$$\lambda = 2{\cdot}12\sqrt{\frac{33{\cdot}5 \times 0{\cdot}148 \times 7{\cdot}62}{U}} \text{ cm.}$$

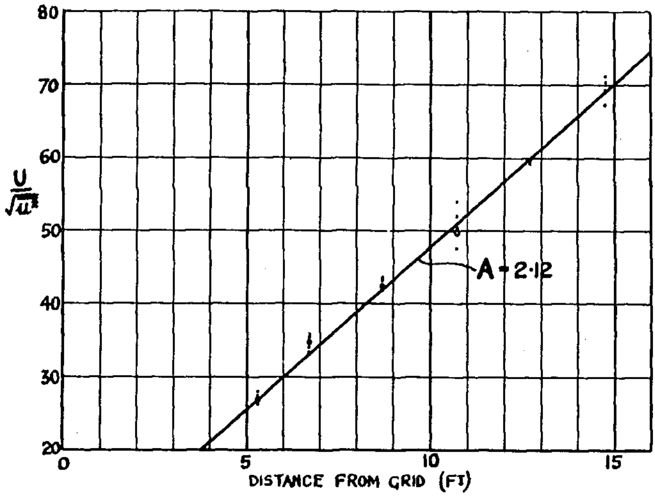


FIG. 5

The values given in column 3, Table II, are calculated from this formula. Comparing the values of λ calculated from the measured dissipation (column 3) with those calculated from the measured spectrum curves using equation (18) (column 2), it will be seen that the agreement is fairly good.

TABLE II. VALUES OF λ

U ft./sec.	λ calculated from spectrum curves	λ calculated from observed dissipation
15	0·86 cm.	0·61 cm.
20	0·63	0·53
35	0·50	0·40

CORRELATION MEASURED WITH BAND FILTER CIRCUITS

Recently Dryden (1937) has made measurements of R_x using various band filter circuits in his amplifier. The action of the band filter is to cut out all disturbances except those whose frequencies lie between certain limits. By supplying truly sinusoidal disturbances of known frequency and amplitude to the filter circuit the characteristic curve showing the response of the circuit to unit input were obtained. If $\phi(n)$ is the ratio of $\overline{u^2}$ measured with the filter to $\overline{u^2}$ measured without it, the characteristic curve $\{\phi(n), n\}$ for one of Dryden's circuits which passes frequencies between 250 and 500 cycles is shown in fig. 6. The measured values of R_x using this filter circuit and with wind speed $U = 20$ ft./sec. are shown in fig. 7.

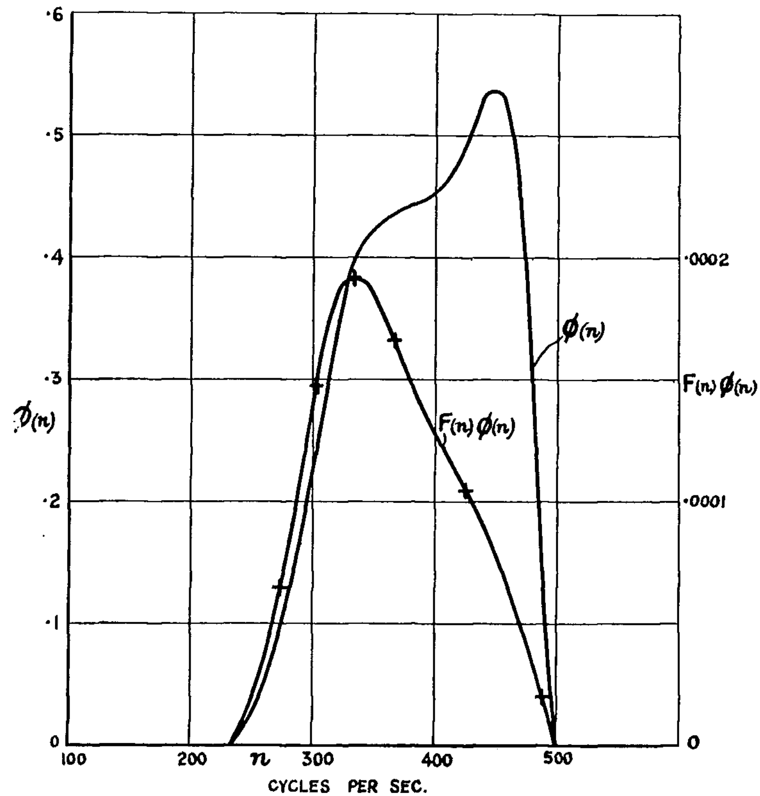


FIG. 6

We have already seen how R_x is related to $F(n)$. It is clear that the same relationship will still hold when the filter circuit is inserted, but $F(n)$ must now be replaced by

$$\frac{F(n)\,\phi(n)}{\int_0^\infty F(n)\,\phi(n)\,dn}.$$

If $R_{x\phi}$ is the value of R_x measured with the band filter circuit ϕ, the formula analogous to (11) is

$$R_{x\phi} = \frac{\int_0^\infty F(n)\,\phi(n)\cos\frac{2\pi nx}{U}\,dn}{\int_0^\infty F(n)\,\phi(n)\,dn}. \tag{22}$$

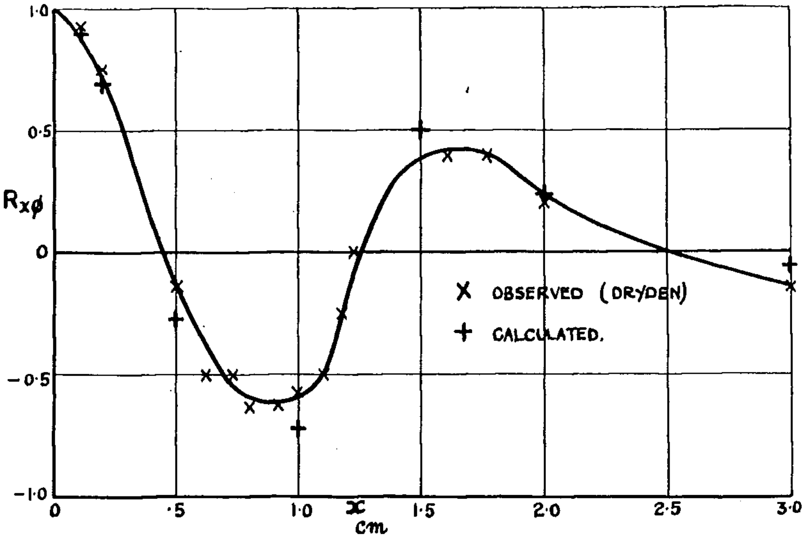


FIG. 7

Before this formula can be used in comparison with Dryden's observed values of $R_{x\phi}$ it is necessary to find $F(n)$. Dryden has not measured $F(n)$ except very roughly, but he has measured R_x in the same air stream and without the filter circuit. His measurements are shown in fig. 8. To calculate $F(n)$ we may use the expression (16) with the observed R_x. The values of $\frac{U}{4}F(n)$ so found is also shown in fig. 8. Taking the values of $F(n)$ from a smooth curve through the calculated points, the values of $F(n)\,\phi(n)$ together can be found for any given values of U and m by multiplying the ordinates of the $F(n)$ and $\phi(n)$ curves. The values of $F(n)\,\phi(n)$ at $U = 20$ ft./sec. ($=610$ cm./sec.) found in this way are shown in fig. 6.

Using a series of values of x, the values of $R_{x\phi}$ have been calculated by numerical integration of (22). The values so calculated are shown in Table III, and are marked in fig. 7. It will be seen that the agreement with Dryden's observations is very good. This agreement provides additional evidence in favour of the main thesis of this paper that R_x and $\frac{U}{2\sqrt{2\pi}} F(n)$ are Fourier transforms of one another, so that each can be predicted when the other has been measured.

TABLE III—CALCULATED VALUES OF $R_{x\phi}$

x (cm.)	0·1	0·2	0·5	1·0	1·5	2·0	3·0
$R_{x\phi}$	+0·91	+0·79	−0·27	−0·72	+0·51	+0·23	−0·06

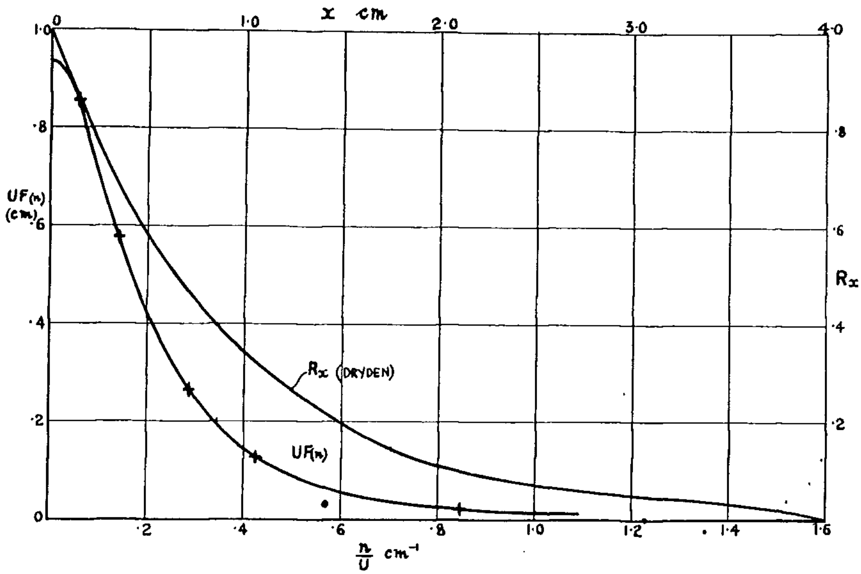


FIG. 8

BEARING OF SPECTRUM MEASUREMENTS ON THEORY OF DISSIPATION

The fact that the $\{UF(n), n/U\}$ curve is independent of U over nearly the whole range indicates that the turbulent flow at a fixed point behind a regular grid is similar, so far as the main features of the flow are concerned, at all speeds. On the other hand the fact that small quantities of very high frequency disturbances appear, and increase as the speed increases, seems to confirm the view frequently put forward by the author that the dissipation of energy is due chiefly to the formation of very small regions where the vorticity is very high. Apart from these very small regions the turbulence behind a grid is similar at all speeds.

Summary

It is shown that a definite connexion exists between the spectrum of the time variation in wind at a fixed point in a wind stream and the curve of correlation between the wind variations at two fixed points. The spectrum curve and the correlation curve are, in fact, Fourier transforms of one another.

As an example of the use of this relationship the spectrum of turbulence in an American wind tunnel was calculated from measurements of correlation by Dryden. In some further experiments Dryden modified this spectrum by inserting a filter circuit and then measured the correlation with this filter in circuit. The modified spectrum is here calculated from the filter characteristics and the Fourier transform theorem is used to calculate the modified correlation curve. The agreement with Dryden's measurements is very good indeed.

The paper ends with some remarks on the bearing of the spectrum measurements on the theory of dissipation of energy in turbulent flow.

References

Dryden, Schubauer, Mock and Skramstad 1937 "Measurements of intensity and scale of wind-tunnel turbulence and their relation to the critical Reynolds number of spheres." National Adv. Comm. Aeronautics, No. 581. (Referred to as Dryden 1937.)

Kármán, T. de 1937 *J. Aero. Sci.* **4**, 131.

Simmons and Salter 1938 *Proc. Roy. Soc.* A. (In the press.)

Taylor, G. I. 1935 *Proc. Roy. Soc.* A, **151**, 421.

A REVIEW OF THE STATISTICAL THEORY OF TURBULENCE*

BY

HUGH L. DRYDEN

National Bureau of Standards

1. Introduction. The irregular random motion of small fluid masses to which the name turbulence is given is of such complexity that there can be no hope of a theory which will describe in detail the velocity and pressure fields at every instant. Existing theories may be classified as either empirical or statistical.

In the empirical theories attention is focused only on the distribution of mean speed and mean pressure, and assumptions are made as to the dependence of the shearing stresses required to satisfy the equations of motion of the mean flow. These assumptions involve one or more empirical constants. While the type of assumption adopted is often selected on the basis of some hypothesis as to the character of the fluctuations of speed and pressure, the theory rests on the final assumption rather than on the hypothesis as to the fluctuations. The various "mixing length" theories are of this type.

In the statistical theories consideration is given to the frequency distribution and mean values of the pressure and of the components of the velocity fluctuations, i.e. to the statistical properties of the fluctuations, and to the relation between the mean motion and these statistical properties.

Some attempts have been made to apply the methods of statistical mechanics of discrete particles. In all such attempts it is necessary to select certain discrete elements corresponding to the particles, and to make some assumption as to the probability of occurrence of various values of associated properties or more directly the frequency distribution of the associated properties. Difficulties are encountered at both points. The best known theory of this type is that of Burgers[1] who selected as elements in two-dimensional flow the points in a square network of equally spaced points and as associated property the value of the stream function. This theory has not as yet led to useful results and is not satisfactory to Burgers himself. Other attempts of

* Received Nov. 19, 1942.

[1] Burgers, J. M., *On the application of statistical mechanics to the theory of turbulent fluid motion*, I to VII, inclusive, Verh. Kon. Akad. v. Wetensch. Amsterdam **32**, 414, 643, 818 (1929); **36**, 276, 390, 487, 620 (1933). Summarized by Trubridge in Reports Phys. Soc. London, 1934, p. 43.

this nature have been made by von Kármán,[3] Noether,[3] Tollmien,[4] Gebelein,[5] Dedebant, Wehrlé and Schereschewsky,[6] and Takahasi.[7]

Many of the statistical theories just mentioned do not require the turbulent fluctuations to satisfy the equations of motion nor do they require the fluid motion to be continuous. A statistical theory of turbulence which is applicable to continuous movements and which satisfies the equations of motion was inaugurated in 1935 by Taylor[8] and further developed by himself and by von Kármán.[9] It is the object of this paper to give a connected account of the present state of this particular statistical theory of turbulence.

2. Turbulent fluctuations and the mean motion. As in other theories of turbulent flow, the flow is regarded as a mean motion with velocity components, U, V, and W, on which are superposed fluctuations of the velocity with components of magnitude u, v, and w at any instant. The mean values of u, v, and w are zero. In most cases U, V, and W are the average values at a fixed point over a definite period of time, although in certain problems it is more convenient to take averages over a selected area or within a selected volume at a given instant. The rules for forming mean values were stated by Reynolds[10] and some further critical discussion by Burgers and others has been recorded in connection with a lecture by Oseen.[11]

When the turbulent motion is produced in a pipe by the action of a constant pressure gradient or near the surface of an object in a wind tunnel in which the fan is operated at a constant speed, there is considerable freedom

---

[3] Kármán, Th. von, *Über die Stabilität der Laminarströmung und die Theorie der Turbulenz*, Proc. 1st Inter. Congr. Appl. Mech., Delft, 1924, p. 97.

[3] Noether, F., *Dynamische Gesichtspunkte zu einer statistischen Turbulenztheorie*, Z. angew. Math. u. Mech. **13**, 115 (1933).

[4] Tollmien, W., *Der Burgersche Phasenraum und einige Fragen der Turbulenzstatistik*, Z. angew. Math. u. Mech. **13**, 331 (1933). Brief abstract of this paper entitled, *On the turbulence statistics in Burgers' phase space*, Physics, **4**, 289 (1933).

[5] Gebelein, H., *Turbulenz: Physikalische Statistik und Hydrodynamik*, Julius Springer, Berlin, 1935.

[6] Dedebant, G., Wehrlé, Ph., and Schereschewsky, Ph., *Le maximum de probabilité dans les mouvements permanents. Application à la turbulence*, Comptes Rendus Ac. Sci. Paris **200**, 203 (1935). Also Dedebant, G., and Wehrlé, Ph., *Sur les équations aux valeurs probables d'un fluide turbulent*, Comptes Rendus Ac. Sci. Paris **206**, 1790 (1938).

[7] Takahasi, K., *On the theory of turbulence*, The Geophysical Magazine **10**, 1 (1936).

[8] Taylor, G. I., *Statistical theory of turbulence*, I–V inclusive, Proc. Roy. Soc. London Ser. A, **151**, 421 (1935) and **156**, 307 (1936). Also, *The statistical theory of isotropic turbulence*, Jour. Aeron. Sci., **4**, 311 (1937).

[9] Kármán, Th. von, *On the statistical theory of turbulence*, Proc. Nat. Acad. Sci. **23**, 98 (1937). Also *The fundamentals of the statistical theory of turbulence*, Jour. Aeron. Sci. **4**, 131 (1937). Also with Howarth, L., *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London Ser. A, **164**, 192 (1938).

[10] Reynolds, O., *On the dynamical theory of incompressible viscous fluids and the determination of the criterion*, Phil. Trans. Roy. Soc. London **186**, 123 (1895).

[11] Oseen, C. W., *Das Turbulenzproblem*, Proc. 3rd Inter. Congr. Appl. Mech., Stockholm, 1931, vol. **1**, p. 3.

in selecting the time interval for which mean values are taken. So long as the time interval is longer than some fixed value dependent on the scale of the apparatus and the speed, the mean values are independent of the magnitude of the time interval selected and there is a clear separation between the turbulent fluctuations and the mean motion. If the mean motion itself is "slowly" variable, as in the case of the natural wind, difficulty arises; the separation becomes imperfect and arbitrary. The slowly variable mean may be taken over time intervals of five minutes, one day, or ten years according to the object of the study and the magnitude of the turbulent fluctuations varies accordingly. Even in flows under constant pressure gradient, there will usually be some experimental difficulty in maintaining the conditions absolutely constant, and the question will naturally arise as to how the fluctuations arising from this source may be eliminated from the "true" turbulent fluctuations.

3. Vortex trails. For a long time every flow in which "fast" fluctuations of velocity occurred was regarded as a turbulent flow but experimental measurements of fluctuations show several identifiable types. The experimental results suggest the limitation of the term "turbulent fluctuation" to one of these types characterized by the random nature of the fluctuations. This random characteristic is in marked contrast with the regularity and periodicity noted in a second type of fluctuation associated with vortex trails.

It is well known that when a cylinder or other object of blunt cross section is exposed to a fluid stream, a vortex trail appears under certain circumstances, vortices breaking away with a regular periodicity. The speed fluctuations observed in the trail are periodic and in themselves do not produce turbulent mixing. At comparatively short distances the regular pattern transforms into an irregular turbulent motion, but the fluctuations within the trail itself do not have the character of the final turbulent fluctuations.

The fluctuations of turbulence are irregular, without definite periodicity with time. The amplitude distribution corresponds to the Gaussian distribution, i.e. the number of times during a long time interval that a given magnitude of fluctuation is reached varies with the magnitude according to the "error" curve.

If this randomness is regarded as an essential feature of the turbulent fluctuations, turbulence is not equivalent to any regular vortex system however complex. The equivalent vortex picture is a large family of vortex systems, whose statistical properties only, not individual histories, are significant.

4. Space and time averages. The speed fluctuations u, v, and w, though designated the fluctuations at a point, are in reality averages throughout a certain volume and over a certain time as are the speed components in the usual hydrodynamic theory. The volume is small in comparison with the dimensions of interest in the flow but large enough to include many molecules. A cube of size 0.001 mm, containing at atmospheric pressure about 2.7×10^7 molecules, satisfies this condition. The time interval is short in comparison with any time interval of interest in the mean properties of the flow but long in comparison with the time required for a molecule to traverse the mean free path. The number of collisions at atmospheric pressure is of the order of 5×10^9 per second and hence a time interval of 10^{-6} seconds would suffice.

No instruments have yet been constructed to give values averaged over so small a volume or so short a time interval. The best performance obtained to date is that of hot wire anemometers which have been developed to the point where average values over a cylindrical volume perhaps 0.01 mm in diameter and 1 mm long and over a time interval of approximately 0.5×10^{-3} seconds can be obtained. Experimental results show that averages over these space and time intervals are not appreciably different from those for somewhat larger space and time intervals and suggest that averages over smaller intervals would not be appreciably different. The results also suggest that measuring equipment that does not approach these space and time intervals gives results which largely reflect the properties of the measuring instrument rather than the properties of the turbulent fluctuations. In other words the measurement is that of a variable mean velocity over space and time intervals fixed by the characteristics of the instrument, rather than measurements of the turbulent fluctuations. If the frequency spectrum of the turbulent fluctuations is known, the effect of the instrument characteristics can be estimated, as discussed in section 19.

5. Pulsations. Reference has previously been made to the difficulty in certain cases of making a clear separation between the mean motion and the turbulent fluctuations, because of the difficulty of defining a time interval long enough to include many fluctuations but small enough so that the mean varies only slowly. The difficulty is often increased by the presence of a fairly rapid variation of the mean speed over large areas, perhaps the entire cross section of the fluid stream, to which the name pulsation may be given. Such a fluctuation is recognizable by the fact that there is a regularity in the space distribution of the fluctuations such that definite phase relations exist. Pulsations have been observed in laminar flow in boundary layers. An essential characteristic of the turbulent fluctuations is an irregularity and randomness in the space distribution as well as in the time distribution.

It is often possible to eliminate the effect of pulsations on the measurements by a low frequency cut-off in the equipment for measuring u, v, and w. The choice of the cut-off frequency is equivalent to a selection of the time interval over which averages are taken to obtain the mean speed and by this device the pulsations are regarded as variations of the mean speed.

6. Continuity of the turbulent motion. It is well known that the structure of a fluid is in the final analysis discontinuous, the fluid consisting of individual molecules. Nevertheless the usual hydrodynamic theory regards the fluid as a continuum. Such an assumption can be justified when the dimensions of the flow system are very large compared to the mean free path of the molecules. The velocity of the fluid at any point is then defined as the vector average of the velocities of the molecules in a small volume surrounding the point, the value obtained being independent of the magnitude and shape of the volume within certain limits.

Some investigators[12] have concluded that the phenomena of turbulence require the assumption of discontinuity in the instantaneous components. The Taylor-von Kármán statistical theory retains the assumption that the fluctuations are continuous functions of space and time as in Reynolds' theory.

The applicability of this assumption is a matter for experimental determination. If experimentally a volume and time interval can be selected which may be regarded as large in comparison with molecular distances and periods but small as compared to the volumes and time intervals of interest in the turbulent fluctuations, the fluctuations may be safely regarded as continuous. As described in section 4, the experimental data perhaps do not prove but do definitely suggest that such a choice is possible and to that extent the assumption of continuity is experimentally justified.

7. The Reynolds stresses. If in the Navier-Stokes equations of motion the components of the velocity are written as $U+u$, $V+v$, $W+w$, thus regarding the motion as a mean motion U, V, W, with fluctuations u, v, w superposed, and mean values taken in accordance with the rules mentioned in section 2, a new set of equations is obtained which differs from the first only in the presence of additional terms added to the mean values of the stresses due to viscosity. These additional terms are called the Reynolds stresses or eddy stresses. The eddy normal stress components are $-\rho\overline{u^2}$, $-\rho\overline{v^2}$, $-\rho\overline{w^2}$ and the eddy shearing stress components are $-\rho\overline{uv}$, $-\rho\overline{vw}$, $-\rho\overline{uw}$. Each stress component is thus equal to the rate of transfer of momentum across the corresponding surface by the fluctuations.

In the light of kinetic theory the eddy stresses closely parallel in origin the viscous stresses. It has been explained how u, v, and w are themselves the mean speeds of many molecules. The effect of the molecular motions appears in the smoothed equations of the continuum as a stress, the components of which are equal to the rate of transfer of momentum by the molecules across the corresponding surfaces.

8. Correlation. If the fluctuations were perfectly random, the eddy shearing stress components $-\rho\overline{uv}$, $-\rho\overline{vw}$, $-\rho\overline{uw}$ would be zero. The existence of eddy shearing stresses is dependent on the existence of a correlation between the several components of the velocity fluctuation at any given point. The coefficient of correlation between u and v is defined as

[12] Kampé de Fériet, J., *Some recent researches on turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass.,1938, p. 352.

$$R_{uv} = \frac{\overline{uv}}{\sqrt{\overline{u^2}}\sqrt{\overline{v^2}}} \cdot \tag{8.1}$$

The mean values $\sqrt{\overline{u^2}}$, $\sqrt{\overline{v^2}}$, and $\sqrt{\overline{w^2}}$ are often called the components of the intensity of the fluctuations.

The eddy shearing stress may be written in terms of the correlation coefficient as

$$-\rho\overline{uv} = -\rho R_{uv}\sqrt{\overline{u^2}}\sqrt{\overline{v^2}} \tag{8.2}$$

and similarly for the other components.

In addition to the correlation between the components of the velocity fluctuations at a given point, the Taylor-von Kármán theory makes much use of correlations between the components of the velocity fluctuations at neighboring points. Denote the components of the fluctuations at one point by u_1, v_1, w_1, and at another point by u_2, v_2, w_2. The coefficient of correlation between u_1 and v_2 is defined as

$$R_{u_1v_2} = \frac{\overline{u_1v_2}}{\sqrt{\overline{u_1^2}}\sqrt{\overline{v_2^2}}} \tag{8.3}$$

and similarly for any other pair. These correlation coefficients form useful tools to describe the statistical properties of the fluctuations with respect to their spatial distribution and phase relationships.

9. Scale of turbulence. The earliest attempt to describe the spatial characteristics of turbulence was the introduction of the mixing length concept, the mixing length being analogous to the mean free path of the kinetic theory of gases. Logical difficulties arise because there are no discrete fluid particles in the turbulent flow which retain their identity. A method of avoiding these difficulties was suggested by Taylor[13] many years ago. He showed that the diffusion of particles starting from a point depends on the correlation R_t between the velocity of a fluid particle at any instant and that of the same particle after a time interval t. If the functional relationship between R_t and t is of such a character that R_t falls to zero at some interval T and remains so for greater intervals, it is possible to define a length l_1 by the relation:

$$l_1 = \sqrt{\overline{v^2}} \int_0^T R_t dt = \sqrt{\overline{v^2}} \int_0^\infty R_t dt \tag{9.1}$$

in which v is the component of the velocity fluctuations transverse to the mean flow and in the direction in which the diffusion is studied.

[13] Taylor, G. I., *Diffusion by continuous movements*, Proc. London Math. Soc. Ser. A, **20,** 196 (1921).

This method of assigning a scale to turbulence is of value in the study of diffusion as described in section 22. It is based on the Lagrangian manner of describing the flow by following the paths of fluid particles. It is more common to use the Eulerian description by considering the stream lines existing in space at any instant. Taylor later[8] suggested a method of describing the scale in the Eulerian system based on the variation of the correlation coefficient R_y between the values of the component u at two points, separated by the distance y in the direction of the y coordinate, as y is varied. The curve of R_y against y represents the statistical distribution of u along the y axis at any instant. If R_y falls to zero and remains zero, a length L may be defined by the relation:

$$L = \int_0^\infty R_y dy. \tag{9.2}$$

The length L is considered a possible definition of the average size of the eddies present and has been found to be a most useful measure of the scale of the turbulence, especially for the case of isotropic turbulence. Correspondingly, a length L_x may be defined by the relation:

$$L_x = \int_0^\infty R_x dx \tag{9.3}$$

where R_x is the correlation between the values of the component u at two points separated by distance x in the direction of the x coordinate.

10. Isotropic turbulence. The simplest type of turbulence for theoretical or experimental investigation is that in which the intensity components in all directions are equal. More accurately, isotropic turbulence is defined by the condition that the mean value of any function of the velocity components and their derivatives at a given point is independent of rotation and reflection of the axes of reference. Changes in direction and magnitude of the fluctuations at a given point are wholly random and there is no correlation between the components of the fluctuations in different directions. Thus $\overline{u^2} = \overline{v^2} = \overline{w^2}$ and $\overline{uv} = \overline{vw} = \overline{uw} = 0$.

There is a strong tendency toward isotropy in all turbulent motions. The turbulence at the center of a pipe in which the flow is eddying or in the natural wind at a sufficient height above the ground is approximately isotropic. A grid of round wires placed in a uniform fluid stream sets up a more or less regular eddy system of non-isotropic character which very quickly transforms into a field of uniformly distributed isotropic turbulence.

The assumption of isotropy introduces many simplifications in the statistical representation of turbulence. The two quantities, intensity and scale, appear to give a description of the statistical properties of the turbulent field which is sufficient for most purposes. Turbulent fields of this type can readily be produced experimentally and studied. The intensity may be varied from less than 0.1 to about 5.0 percent of the mean speed and the scale independently from a few mm to 25 mm.[14]

11. Decay of isotropic turbulence. The kinetic energy of the turbulent fluctuations per unit volume is equal to $\frac{1}{2}\rho(\overline{u^2}+\overline{v^2}+\overline{w^2})$ which for isotropic turbulence becomes $(3/2)\rho\overline{u^2}$. The rate of decay is therefore $-(3/2)\rho d(\overline{u^2})/dt$. If the isotropic turbulence is superposed on a stream of uniform speed U, we may write $dt=dx/U$ and hence the rate of decay with respect to distance x as $-(3/2)\rho U d(\overline{u^2})/dx$.

In a fully developed turbulent flow the Reynolds stresses are proportional to the squares of the turbulent fluctuations. The work done against these stresses, which in the absence of external forces must come from the kinetic energy of the system, is proportional to $\rho u'^3/L$ where u' is written for $\sqrt{\overline{u^2}}$ and L is a linear dimension defining the scale of the system, which may be taken as the L defined by (9.2). Equating the two expressions for the dissipation and designating the constant of proportionality as $3\,A$, we find:

$$-(3/2)\rho U d(u'^2)/dx = 3A\rho u'^3/L \tag{11.1}$$

or

$$L d(U/u')/dx = A. \tag{11.2}$$

Integrating:

$$U/u' - U/u_0' = A\int_{x_0}^{x} dx/L \tag{11.3}$$

where U/u_0' is the value of U/u' at $x=x_0$. This equation has been found to give a very good representation of the experimental data. The essential features of the derivation were given by Taylor. To evaluate the integral, L must be known as a function of x. Taylor's first proposal was to assume that L is independent of x and proportional to the mesh M of the grid giving rise to the turbulence. If L is constant,

$$U/u' - U/u_0' = A(x - x_0)/L \tag{11.4}$$

giving a linear variation of U/u' with x. Assuming $L/M=k$, Taylor found values of A/k for data from various sources varying between 1.03 and 1.32.

---

[14] Dryden, H. L., Schubauer, G. B., Mock, W. C., Jr., and Skramstad, H. K., *Measurements of intensity and scale of wind-tunnel turbulence and their relation to the critical Reynolds number of spheres*, Tech. Rept. Nat. Adv. Comm. Aeron. No. 581 (1937).

When measured values of L became available it was found that L increased as x increased, the results being represented empirically within the accuracy of the measurements by the relation $L = L_0 + c(x - x_0)$, whence

$$U/u' - U/u_0' = (A/c) \log_e [1 + c(x - x_0)/L_0]. \qquad (11.5)$$

Taylor[15] found values of A for data from various sources varying between 0.43 and 0.19.

Further study suggests another relation for the variation of L with x. A discussion of the general theory will be deferred until section 17 and the question discussed on purely dimensional considerations. If one assumes that du'/dt, the rate of change of intensity, and dL/dt, the rate of change of scale, are determined solely by the values of L and u', i.e. that viscosity and upstream conditions have no influence, it follows from dimensional reasoning that

$$Ld(1/u')/dt = A \quad \text{and} \quad (1/u')dL/dt = B \qquad (11.6)$$

or

$$Ld(U/u')/dx = A \quad \text{and} \quad (U/u')dL/dx = B \qquad (11.7)$$

where A and B are numerical constants. The first equation of each pair is the same as equation (11.2); the second is a new relation.

Integration of equations (11.6) and (11.7) leads to the relations:

$$\frac{u_0'}{u'} = \left[1 + \frac{(A + B)u_0'(x - x_0)}{L_0 U}\right]^{A/(A+B)} \qquad (11.8)$$

and

$$\frac{L}{L_0} = \left[1 + \frac{(A + B)u_0'(x - x_0)}{L_0 U}\right]^{B/(A+B)} \qquad (11.9)$$

where u_0' and L_0 are the values at $x = 0$.

If it is desired to introduce a reference dimension pertaining to the dimensions of the grid producing the disturbance, this may be done, but according to equations (11.8) and (11.9) any dimension may be used and the decay does not depend on its value. The mesh distance M is often used but certain results reported by von Kármán[16] show that if M/d is not too small, the use

[15] Taylor, G. I., *Some recent developments in the study of turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 294 (1938). See later detailed report of measurements in Hall, A. A., *Measurements of the intensity and scale of turbulence*, Rept. and Memo. No. 1842, Aeronautical Research Committee, Great Britain (1938).

[16] Kármán, Th. von, *Some remarks on the statistical theory of turbulence*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 1938, p. 347. The grid dimensions are not given in

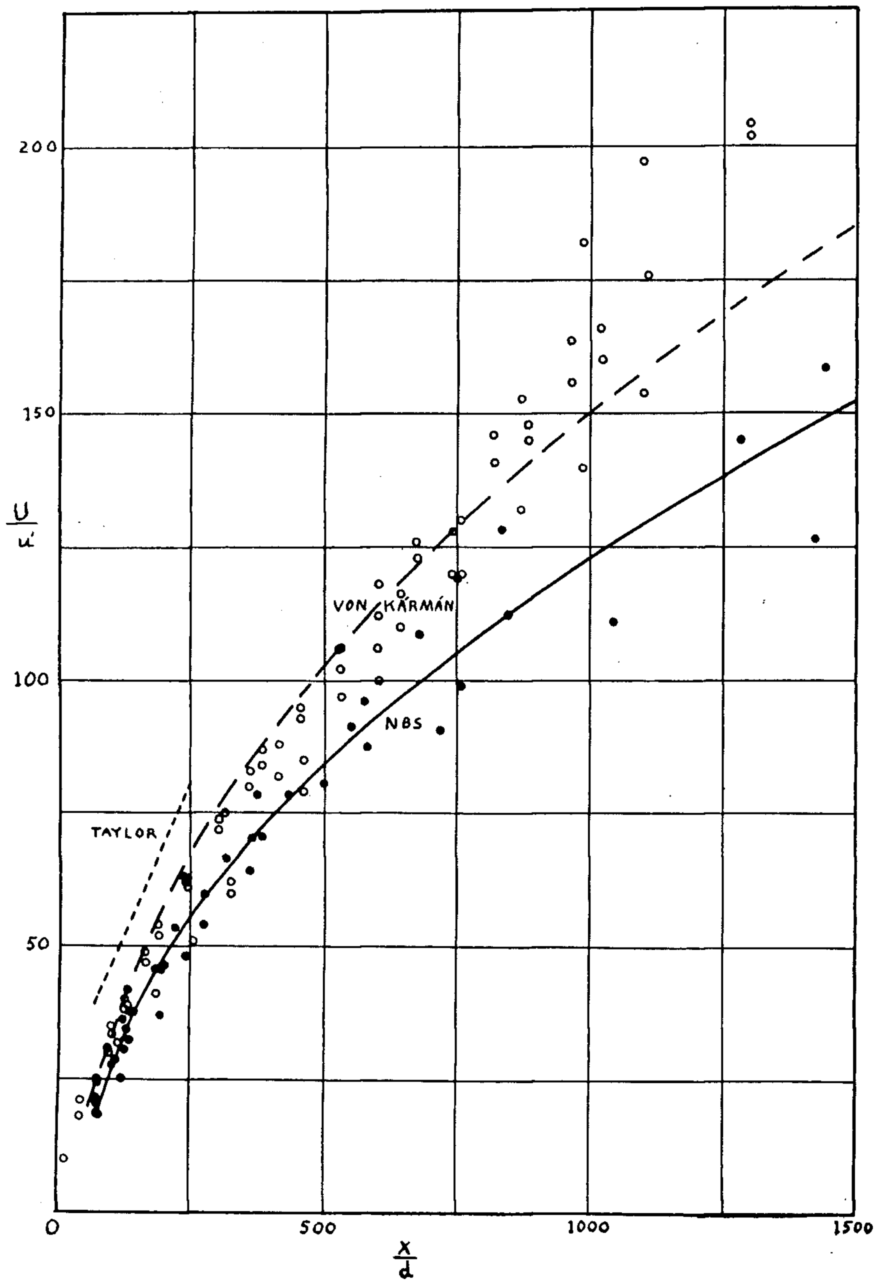


FIG. 1. The turbulent fluctuation u' behind a grid of wires of diameter d as a function of distance x from the grid.

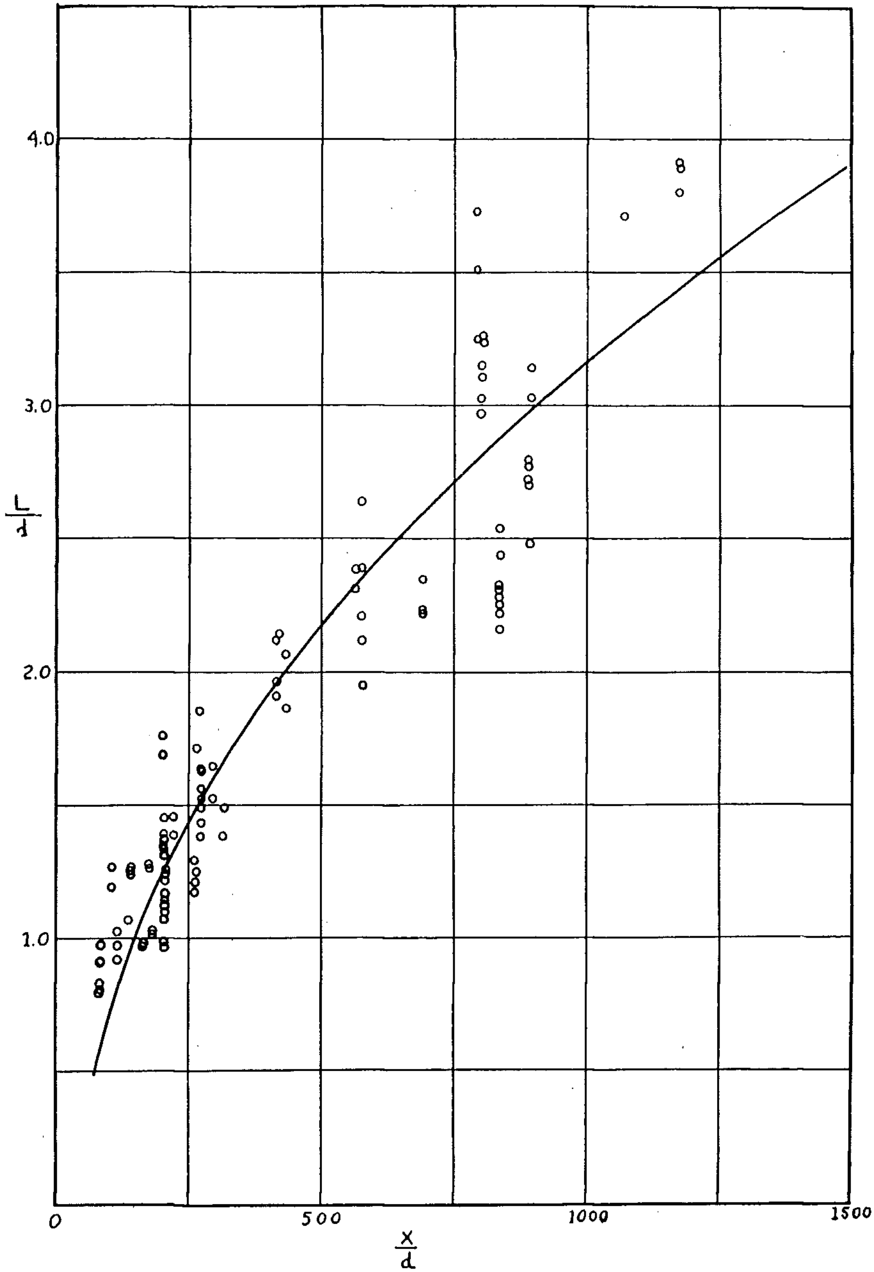


FIG. 2. The scale L behind a grid of wires of diameter d as a function of distance x from the grid.

of the wire diameter d as the reference dimension leads to a single curve for all grids irrespective of the mesh-diameter ratio.

The available data are plotted in Figs. 1 and 2 from references in footnotes 14, 15, and 16. The solid curves are respectively

$$(U/u')^2 = 400[(1 + 0.04(x/d - 80)] \tag{11.10}$$

and

$$(L/d)^2 = 0.264[(1 + 0.04(x/d - 80)] \tag{11.11}$$

which are in the form of equations (11.8) and (11.9) with the constants $A = B = 0.2056$. These curves are frankly selected to fit the National Bureau of Standards data.

If one considers the complete system of screen and turbulent field, dimensional considerations suggest that for geometrically similar screens whose scale is fixed by some characteristic dimension, such as the mesh length M, the ratios u'/U and L/M would be a function of x/M, of the Reynolds Number UM/ν and of the turbulence of the free stream u_1'/U, in which the screen is placed. If the screens are not geometrically similar but are made up of cylindrical rods of diameter d, the intensity and scale also depend on d/M and on the roughness of the screen. The effects of these parameters have not been fully investigated, and doubtless a part of the discrepancy between the available results is to be ascribed to the influence of these factors.

For example, the screens used at the National Bureau of Standards were either woven wire screens or wooden screens with fairly rough surfaces with the members interlacing in the wire screens and intersecting in the wooden screens. The ratio d/M varied from 0.186 to 0.201. The screens used by Hall were arranged in two planes, i.e., horizontal rods in one plane, vertical rods just touching the horizontal rods but in another plane. The ratio d/M was 0.184 to 0.188. Von Kármán has studied the effect of varying d/M from 0.086 to 0.462 and has used screens both of the woven type (results published by von Kármán, loc. cit.) and of the biplane type (results not published). A study of these data suggests that the difference between the results for woven screens and biplane screens is unimportant and that if results are plotted in terms of x/d rather than x/M the effect of d/M is small for values of d/M near 0.2. No data are available on the effect of roughness.

Few data are available on the effect of free stream turbulence. Hall obtained an increase of about 10 to 20 percent in u' for a 1-inch screen at the

the paper, but Professor von Kármán has kindly supplied them as follows:

Grid	*Mesh Distance, M inches*	*Wire Diameter, d inches*	*M/d*
1	4.96	0.230	2.16
2	5.00	.105	4.75
3	5.07	.084	6.03
4	4.99	.043	11.6

same value of x/M by increasing the free stream turbulence from 0.2 percent to 1.3 percent. We have had the opportunity of making some measurements behind the same 1-inch screen used in the measurements described in NACA Technical Report No. 581 in an airstream for which the free stream turbulence is 0.03 percent as compared with 0.85 percent for the older measurements. The results are shown in Fig. 3 as compared with Hall's measurements. It is obvious that the turbulence of the free stream is one of the controlling factors, but not the only one.

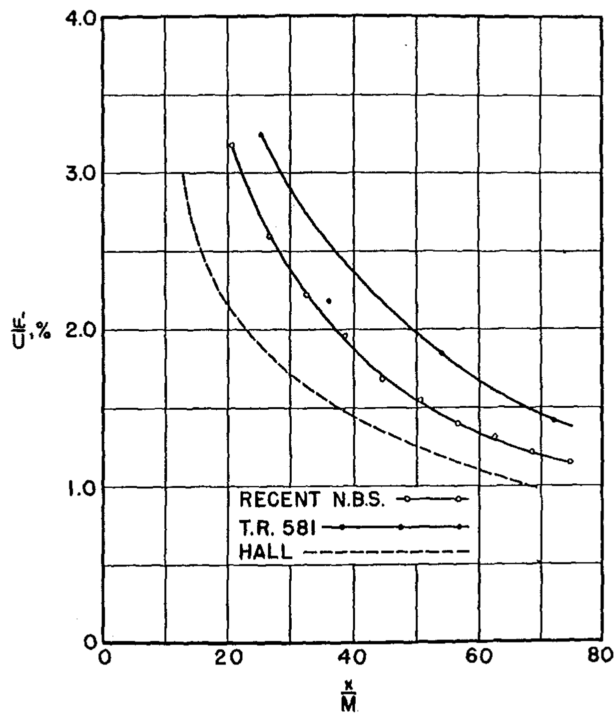


FIG. 3. Effect of free stream turbulence on the turbulence behind a 1-inch screen.

The study of the turbulent field behind screens as affected by numerous parameters is of interest from the standpoint of a study of screens. However, the turbulent field may be regarded from another point of view, i.e. in relation solely to the theory of isotropic turbulence. If the turbulence is truly isotropic, and if its characteristics can be adequately described by the two quantities, intensity and scale, its behavior can depend only on the values of intensity and scale at some given point. The details of construction of the source screen and its distance upstream are of no importance. Even the influence of the turbulence of the free stream should be absorbed in the given values of u' and L at some one point. The decay of isotropic turbulence is considered from this point of view in section 17.

12. Effect of contraction. The behavior of turbulence in a contracting stream is of interest in connection with the flow in the entrance cone of a wind tunnel. Prandtl[17] suggested that the longitudinal components of the fluctuations were reduced in the ratio of 1 to l where l is the ratio of the entrance area to the exit area of the cone. This result was derived on the assumption that the gain in energy is the same for all filaments traversing the cone. The same result was obtained from the Helmholtz vortex theorem, which was also used to show that the lateral components were increased in the ratio $\sqrt{l}$. Since the mean speed increases proportional to l, the values of u'/U and v'/U are reduced according to this theory in the ratios $1/l^2$ and $1/\sqrt{l}$ respectively. This computation neglects the decay of the turbulence because of viscosity.

Taylor[18] computed the effect of a contraction on certain mathematically defined forms of disturbance. Two objections may be offered to this treatment. First, as in Prandtl's treatment, the decay of the turbulence is neglected. Second, the computation is made on a regular disturbance which is assumed to retain its regularity. When the rapid development of an isotropic turbulent field from a Kármán vortex trail is considered, it is hard to believe that a regular vortex pattern could retain its character throughout the length of a wind tunnel entrance cone unless the scale was very large indeed.

If it is assumed that the istropic turbulent field is unaffected by changes in the mean speed, the decrease in u' may be computed from the decay during the time required for the fluid to traverse the cone. This time interval is $\int_{x_0}^{x_1} dx/U$. If A is the area of the cross section at any value of x, $UA = U_0A_0$ where U_0 and A_0 are the values at $x = x_0$, and hence the time interval is $\int_{x_0}^{x_1} A\ dx/U_0A_0$.

There are as yet no suitable experimental data for checking any theory. In the measurements quoted by Taylor all the data were obtained sufficiently close to a grid to lie within the non-isotropic turbulence of the vortex trails from the individual wires.

13. The correlation tensor function. Von Kármán[19] introduced the correlation tensor function in the statistical theory of turbulence as a generalization of the particular correlation coefficients discussed by Taylor. The correlation coefficients between any component of the speed fluctuation at a given point and any component of the speed fluctuation at another point

---

[17] Prandtl, L., *Herstellung einwandfreier Luftströme (Windkanäle)*, Handbuch der Experimentalphysik, F. A. Barth, Leipzig, 1932, Vol. **4**, Part 2, p. 73.

[18] Taylor, G. I., *Turbulence in a contracting stream*, Z. angew. Math. u. Mech. **15**, 91 (1935).

[19] Kármán, Th. von, and Howarth, L., *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London, Ser. A, **164**, 192 (1938).

form a tensor. If one point is held fixed and the other varied, the tensor varies as a function of the coordinates of the variable point with respect to the fixed point. We may speak of this function as the correlation tensor function.

In isotropic turbulence the correlation tensor has spherical symmetry and the several components are functions only of the distance r between the two points, and of the time t. Denote by u_1, v_1, w_1 and u_2, v_2, w_2 the components of the velocity fluctuations at the two points P_1 and P_2 having coordinates $(x_1, 0, 0)$ and $(x_2, 0, 0)$ respectively. Suppose that $\overline{u_1^2}$, $\overline{v_1^2}$, $\overline{w_1^2}$, which by isotropy are equal, are independent of position and equal to $\overline{u_2^2}$. Then $\overline{u_2^2}=\overline{v_2^2}=\overline{w_2^2}=\overline{u^2}$.

The correlation coefficients $\overline{v_1v_2}/\overline{u^2}$ and $\overline{w_1w_2}/\overline{u^2}$ will be identical because of isotropy and will be some particular function of the distance r between

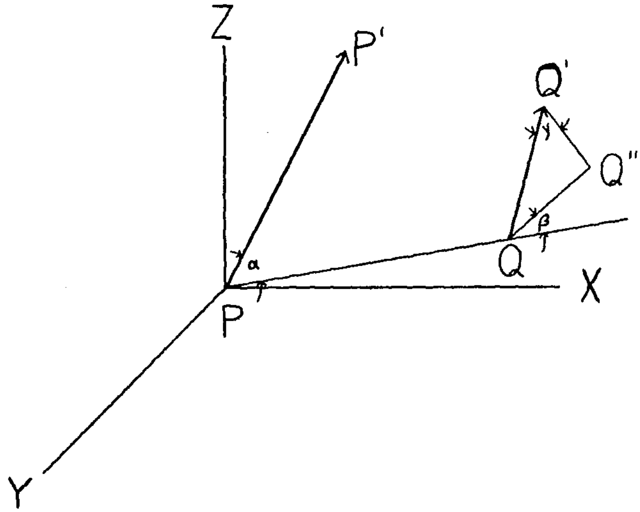


FIG. 4.

P_1 and P_2 and of the time t, say $g(r, t)$. The correlation coefficient $\overline{u_1u_2}/\overline{u^2}$ will also be a function of r and t, say $f(r, t)$. The correlation coefficients $\overline{u_1v_2}/\overline{u^2}$, $\overline{u_1w_2}/\overline{u^2}$, $\overline{v_1u_2}/\overline{u^2}$, $\overline{v_1w_2}/\overline{u^2}$, $\overline{w_1u_2}/\overline{u^2}$ and $\overline{w_1v_2}/\overline{u^2}$ can be shown to be zero. Thus if the Y and Z axes are rotated about the X axis through 180°, the absolute values of all components are unchanged but the signs of the v and w components are reversed. Denoting values referred to the new axes by capital letters, $U_1=u_1$, $U_2=u_2$, $V_1=-v_1$, $V_2=-v_2$, $W_1=w_1$, $W_2=-w_2$, so that, for example, $\overline{U_1V_2}=-\overline{u_1v_2}$. But by isotropy, the value of any function of the components is unchanged by rotation of the axes, and therefore $\overline{U_1V_2}=\overline{u_1v_2}$. To satisfy both relations $\overline{u_1v_2}$ must equal zero. Similarly for the other terms containing u_1 or u_2. By reflection in the XZ plane $\overline{v_1w_2}$ and $\overline{w_1v_2}$ may likewise be shown to be zero.

The correlation coefficient for components of the fluctuations in any arbitrary directions at any two points may be expressed in terms of the functions $f(r, t)$ and $g(r, t)$ and the geometrical parameters. Consider any two points P and Q and components of the fluctuations p in the direction PP′ at P and q in the direction QQ′ at Q. (Fig. 4). Denote by QQ″ the orthogonal projection of QQ′ on the plane PP′Q; by α, β, and γ the angles P′PQ, $\pi -$PQQ″, and QQ′Q″; and by p_1, p_2, p_3 and q_1, q_2, q_3 the components of the fluctuations at P and Q in the direction PQ, in the direction normal to PQ and Q′Q″, and in the direction Q″Q′. Then

$$p = p_1 \cos\alpha + p_2 \sin\alpha$$

$$q = q_1 \cos\beta \sin\gamma + q_2 \sin\beta \sin\gamma + q_3 \cos\gamma. \qquad (13.1)$$

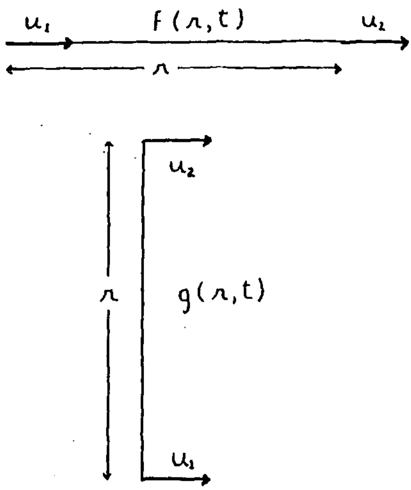


FIG. 5. The principal double correlations in isotropic turbulence.

Hence

$$\overline{pq} = \overline{p_1 q_1} \cos\alpha \cos\beta \sin\gamma + \overline{p_2 q_2} \sin\beta \sin\alpha \sin\gamma \qquad (13.2)$$

the other terms vanishing as proved in the preceding paragraph. In terms of $f(r, t)$ and $g(r, t)$

$$\overline{pq}/\overline{u^2} = [f(r, t) \cos\alpha \cos\beta + g(r, t) \sin\alpha \sin\beta] \sin\gamma. \qquad (13.3)$$

The correlations denoted by $f(r, t)$ and $g(r, t)$ are indicated in Fig. 5.

If now any two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) and speed fluctuations with components u_1, v_1, w_1 and u_2, v_2, w_2 are considered, the nine quantities $\overline{u_1u_2}$, $\overline{u_1v_2}$, $\overline{u_1w_2}$, $\overline{v_1u_1}$, $\overline{v_1v_2}$, $\overline{v_1w_2}$, $\overline{w_1u_1}$, $\overline{w_1v_2}$, and $\overline{w_1w_2}$ are the components of a second rank tensor. Each one may be evaluated by equation (13.2) in terms of $f(r, t)$ and $g(r, t)$ with the result in tensor notation

$$\boldsymbol{R} = \frac{f(r, t) - g(r, t)}{r^2} \boldsymbol{rr} + g(r, t)\boldsymbol{I} \qquad (13.4)$$

where $\boldsymbol{r}$ is the vector having components $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$

r is $|\boldsymbol{r}|$ and $\boldsymbol{I}$ is the unit tensor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

The velocity fluctuations satisfy the equation of continuity. Hence

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} + \frac{\partial w_2}{\partial z_2} = 0. \tag{13.5}$$

Multiplying by $u_1/\overline{u^2}$ which is independent of x_2, y_2, z_2 and introducing the correlation coefficients $R_{u_1u_2}$, etc. and the components X, Y, Z of $\boldsymbol{r}$:

$$\frac{\partial R_{u_1u_2}}{\partial X} + \frac{\partial R_{u_1v_2}}{\partial Y} + \frac{\partial R_{u_1w_2}}{\partial Z} = 0. \tag{13.6}$$

From equation (13.4)

$$R_{u_1u_2} = \frac{f-g}{r^2} X^2 + g; \qquad R_{u_1v_2} = \frac{f-g}{r^2} XY; \qquad R_{u_1w_2} = \frac{f-g}{r^2} XZ$$

whence, remembering that $X^2+Y^2+Z^2=r^2$, $\partial r/\partial X = X/r$, $\partial f/\partial X = (\partial f/\partial r)(\partial r/\partial X) = (X/r)(\partial f/\partial r)$, etc., equation (13.6) becomes

$$X[2(f-g) + r(\partial f/\partial r)] = 0. \tag{13.7}$$

The continuity equation must be true for any value of X. Hence

$$2f(r, t) - 2g(r, t) = -r\partial f(r, t)/\partial r. \tag{13.8}$$

The correlation tensor can thus be expressed in terms of a single scalar function, either $f(r, t)$ or $g(r, t)$. The function $g(r, t)$ is the correlation coefficient previously denoted by R_y. The scale $L = \int_0^\infty R_y\, dy = \int_0^\infty g\, dr$. The integral $\int_0^\infty R_x\, dx = \int_0^\infty f\, dr$ is termed the longitudinal scale L_x to distinguish it from the lateral scale L. Obviously from equation (13.8)

$$L - L_x = \tfrac{1}{2}\int_0^\infty r(\partial f/\partial r)dr = \tfrac{1}{2}\int_0^\infty x(\partial R_x/\partial x)dx. \tag{13.9}$$

Since f and g are even functions of r,

$$f = 1 + f_0'' r^2/2 + \cdots \tag{13.10}$$

$$g = 1 + g_0'' r^2/2 + \cdots . \tag{13.11}$$

From equation (13.8), $2f_0'' = g_0''$, whence for small values of r,

$$\begin{aligned} \boldsymbol{R} &= [1 + (g_0''/2)r^2]\boldsymbol{I} + [(f_0'' - g_0''/2]\boldsymbol{rr} \\ &= (1 + f_0'' r^2)\boldsymbol{I} - (1/2)f_0''\boldsymbol{rr}. \end{aligned} \tag{13.12}$$

We require later the second derivatives of $\boldsymbol{R}$ at $r=0$, i.e. $X=Y=Z=0$, as follows:

$$\frac{\partial^2 R_{u_1u_2}}{\partial X^2} = \frac{\partial^2 R_{v_1v_2}}{\partial Y^2} = \frac{\partial^2 R_{w_1w_2}}{\partial Z^2} = f_0'' \tag{13.13}$$

$$\frac{\partial^2 R_{u_1u_2}}{\partial Y^2} = \frac{\partial^2 R_{u_1u_2}}{\partial Z^2} \text{ and similar terms obtained by cyclic exchange } = 2f_0'' \quad (13.14)$$

$$\frac{\partial^2 R_{u_1v_2}}{\partial X \partial Y} \text{ and similar terms obtained by cyclic exchange } = -(1/2)f_0''. \quad (13.15)$$

All others, e.g. $\frac{\partial^2 R_{u_1v_2}}{\partial X \partial Z}$ etc. are zero. (13.16)

Von Kármán points out that the correlation tensor is of the same form as the stress tensor for a continuous medium when there is spherical symmetry. In the analogy $f(r)$ corresponds to the principal radial stress at any point, $g(r)$ to the principal transverse stress, and the several R's to the stress components over planes normal to the coordinate axes. The relation between f and g given by the continuity equation corresponds to the condition for equilibrium of the stresses.

Equation (13.8) has been experimentally checked at the National Physical Laboratory.[15]

14. Correlation between derivatives of the velocity fluctuations. In further developments it will be necessary to know the mean values of the products of the derivatives of the components of the fluctuations at a given point, for example $\overline{(\partial u_1/\partial x_1)(\partial v_1/\partial y_1)}$. These mean values may readily be computed from the correlation tensor. Thus:

$$\frac{\partial(\overline{u_1 v_2})}{\partial x_1} = \overline{u^2}\frac{\partial(R_{u_1v_2})}{\partial x_1} = -\overline{u^2}\frac{\partial(R_{u_1v_2})}{\partial X} \quad (14.1)$$

Since v_2 is not a function of x_1, this may be written

$$\overline{(\partial u_1/\partial x_1)v_2} = -\overline{u^2}\partial R_{u_1v_2}/\partial X. \quad (14.2)$$

Differentiating now with respect to y_2

$$\overline{\frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial y_2}} = \frac{\partial}{\partial y_2}\left(\overline{\frac{\partial u_1}{\partial x_1}v_2}\right) = -\overline{u^2}\frac{\partial}{\partial y_2}\left(\frac{\partial R_{u_1v_2}}{\partial X}\right) = -\overline{u^2}\frac{\partial^2 R_{u_1v_2}}{\partial X \partial Y}. \quad (14.3)$$

Now letting P_1 and P_2 coincide,

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = -\overline{u^2}\left(\frac{\partial^2 R_{uv}}{\partial X \partial Y}\right)X = Y = 0. \quad (14.4)$$

The limiting value of the second derivative has previously been computed (equation (13.15)), whence

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = \frac{f_0''}{2}\overline{u^2}. \quad (14.5)$$

By similar reasoning it may be shown that

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} = \overline{\left(\frac{\partial w}{\partial z}\right)^2} = -\overline{u^2}f_0'' \tag{14.6}$$

$$\begin{aligned}\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial u}{\partial z}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial z}\right)^2} = \overline{\left(\frac{\partial w}{\partial x}\right)^2}\\ = \overline{\left(\frac{\partial w}{\partial y}\right)^2} = -2\overline{u^2}f_0\end{aligned} \tag{14.7}$$

and

$$\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} = \overline{\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}} = \overline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}} = \frac{\overline{u^2}}{2}f_0''. \tag{14.8}$$

The method can be extended to derivatives of higher order.

15. Triple correlations. Von Kármán designates the mean values of the product of three components of the velocity fluctuations, two of which are taken at one arbitrary point and the third at a second arbitrary point, as triple correlations. They arise when correlation coefficients are introduced into the equations of motion. He shows that the triple correlations are components of a tensor of third rank designated T which is a function of X, Y, Z and the time. He proves that in isotropic turbulence this tensor can be expressed in terms of three functions $h(r, t)$, $k(r, t)$, and $q(r, t)$ corresponding to the correlations shown in Fig. 6, and that the development of these functions in powers of r begins with the r^3 term. The equation of continuity permits the expression of k and q in terms of h by the relations:

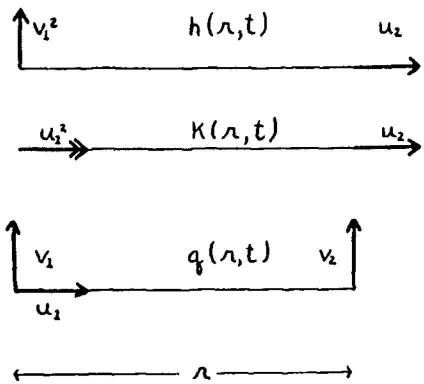


FIG. 6. The principal triple correlations in isotropic turbulence.

$$k = -2h \tag{15.1}$$

$$q = -h - (r/2)(dh/dr). \tag{15.2}$$

Thus the tensor T can be expressed in terms of a single scalar function $h(r, t)$.

16. Propagation of the correlation with time. The fluctuations are assumed to satisfy the equations of motion, namely,

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} + w_1\frac{\partial u_1}{\partial z_1} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial y_1^2} + \frac{\partial^2 u_1}{\partial z_1^2}\right) \tag{16.1}$$

and the two equations obtained by cyclic permutation.

Multiplying this equation by u_2, introducing X, Y, and Z, and taking mean values:

$$\overline{u_2 \frac{\partial u_1}{\partial t}} - \frac{\partial(\overline{u_1^2 u_2})}{\partial X} - \frac{\partial(\overline{u_1 v_1 u_2})}{\partial Y} - \frac{\partial(\overline{u_1 w_1 u_2})}{\partial Z}$$

$$= -\frac{1}{\rho}\overline{u_2 \frac{\partial p}{\partial x_1}} + \nu\left(\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right). \tag{16.2}$$

By an analogous procedure, it may be shown that:

$$\overline{u_1 \frac{\partial u_2}{\partial t}} - \frac{\partial \overline{u_2^2 u_1}}{\partial X} - \frac{\partial \overline{u_2 v_2 u_1}}{\partial Y} - \frac{\partial \overline{u_2 w_2 u_1}}{\partial Z}$$

$$= -\frac{1}{\rho}\overline{u_1 \frac{\partial p}{\partial x_2}} + \nu\left(\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right). \tag{16.3}$$

Von Kármán shows that the pressure terms vanish. Adding the two equations and introducing the correlation coefficients, we find

$$\frac{\partial}{\partial t}(\overline{u^2} R_{u_1 u_2}) - (\overline{u^2})^{3/2}\frac{\partial}{\partial X}(\overline{u_1^2 u_2} + \overline{u_2^2 u_1}) - (\overline{u^2})^{3/2}\frac{\partial}{\partial Y}(\overline{u_1 v_1 u_2} + \overline{u_2 v_2 u_1})$$

$$- (\overline{u^2})^{3/2}\frac{\partial}{\partial Z}(\overline{u_1 w_1 u_2} + \overline{u_2 w_2 u_1}) \tag{16.4}$$

$$= 2\nu\overline{u^2}\left[\frac{\partial^2 \overline{u_1 u_2}}{\partial X^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Y^2} + \frac{\partial^2 \overline{u_1 u_2}}{\partial Z^2}\right]$$

This equation may be expressed in terms of the functions f, g, k, q and h. Then by using the relations (15.1) and (15.2) between these functions obtained from the equation of continuity, a partial differential equation between f and h is obtained, namely

$$\frac{\partial(f\overline{u^2})}{\partial t} + 2(\overline{u^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{16.5}$$

This is the equation for the change of the function f with time, but it cannot be solved without some knowledge of the function h.

17. Self-preserving correlation functions. Let us suppose that the functions $f(r, t)$ and $h(r, t)$ preserve the same form as t increases, only the scale varying. Such functions will be termed "self-preserving." If L is some measure of the scale of the correlation curve, f and h will be functions of r/L only, where L is a function of t. The length L may be any measure of the scale such as the radius of curvature of the correlation curve at $r=0$ or any other designated point, the value of r for a given value of the correlation coefficient, or the quantity obtained by integration of the correlation coefficient from $r=0$ to infinity which has previously been termed the scale of the turbulence. Introducing the new variable $\psi=r/L$ and placing $(\overline{u^2})^{1/2}=u'$ in equation (16.5), we obtain

$$\frac{fL}{u'^3}\frac{du'^2}{dt}-\frac{1}{u'}\frac{dL}{dt}\psi\frac{\partial f}{\partial\psi}+2\left(\frac{\partial h}{\partial\psi}+4\frac{h}{\psi}\right)=\frac{2}{N}\left(\frac{\partial^2 f}{\partial\psi^2}+\frac{4}{\psi}\frac{\partial f}{\partial\psi}\right) \quad (17.1)$$

where N is the Reynolds Number of the turbulence $u'L/\nu$. Since the coefficient of the third term is a numerical constant, the functions f and h will be functions of ψ and t alone only if the coefficients of the other terms are also numerical constants. This requires that

$$\frac{L}{u'^3}\frac{du'^2}{dt}=-L\frac{d(1/u')}{dt}=-A \quad (17.2)$$

$$\frac{1}{u'}\frac{dL}{dt}=B \quad (17.3)$$

$$\frac{u'L}{\nu}=N_0 \quad (17.4)$$

where A, B, and N_0 are independent of u', L, and t. It is readily shown that these relations are consistent only if $A=B$ and that the solutions are

$$\frac{1}{u'^2}-\frac{1}{u_0'^2}=\frac{2A}{N_0\nu}t \quad (17.5)$$

$$L^2-L_0^2=2AN_0\nu t \quad (17.6)$$

where u_0' and L_0 are the values for $t=0$ and $u_0'L_0/\nu=N_0$.

These equations are in the form of equations (11.8) and (11.9) with $A=B$ and agree well with the formulation of the experimental data represented by equations (11.10) and (11.11) with $2AN_0\nu/L_0^2=2Au_0^2/N_0\nu=2A\sqrt{u_0^2}/L_0=0.04$, corresponding to $t=0$ at a distance of 80 wire diameters from the grid. The constant A is equal to 0.2056 when L is defined as $\int_0^\infty R_y dy$.

For self-preserving turbulence equation (16.5) becomes

$$-Af-A\psi(\partial f/\partial\psi)+2(\partial h/\partial\psi+4h/\psi)=(2/N_0)\left[(\partial^2 f/\partial\psi^2+(4/\psi)(\partial f/\partial\psi)\right]. \quad (17.7)$$

This equation determines the shape of the correlation curve. Von Kármán[19] discusses the shape when the function h is neglected. The shape depends on the Reynolds Number N_0 of the turbulence. The shape also depends on the constant A but closer examination shows that A is always associated with L

and is dependent on the method of defining L. If ψ is set equal to rA/L instead of r/L, and the length L/A is used instead of L in the definition of the Reynolds Number of the turbulence, the A disappears from equation (17.7). Whether the values of L defined by $\int g dr$ will yield the same values of A for all shapes of correlation curves described by (17.7) cannot be definitely answered.

Approximate solutions of (17.7) are not easy since it turns out that f varies with N_0 in such a manner that, for small values of ψ at least, the term on the right-hand side is of the order of unity.

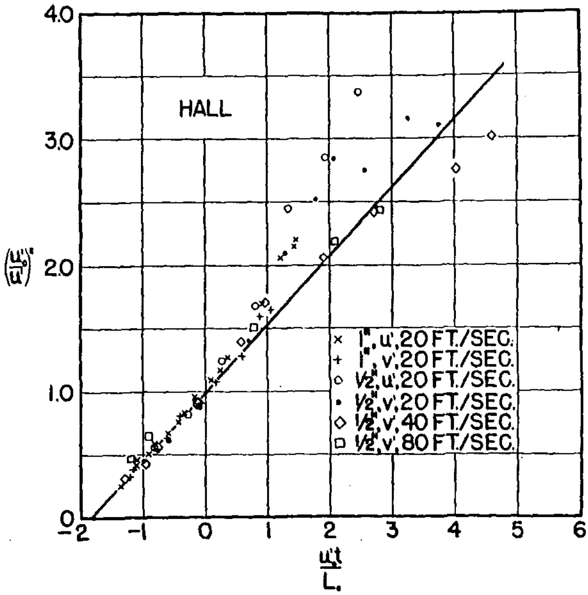


FIG. 7. Hall's measurements of turbulence behind screens.

According to this suggested theory, the shape is self-preserving and the Reynolds Number remains constant during the decay of a given turbulent field. The scale approaches very large values as the intensity approaches very small values. The length λ (which is discussed in section 18) is proportional to L. For different values of the Reynolds Number of the turbulence the constant of proportionality varies inversely as the square root of the Reynolds Number. Likewise the shape of the correlation curve varies with the Reynolds number of the turbulence.

Equation (17.6) shows the same functional relation between the scale and the time as given by Prandtl at the Turbulence Symposium as a result of his analysis of photographs of the decay of isotropic turbulence.

Von Kármán also discusses the case in which the assumption is made that the self-preserving feature applies only to large values of ψ and the Reynolds Number N_0 is sufficiently large that the right-hand term of (17.7) can be neglected. In this case (17.2) and (17.3) are obtained without (17.4) and the solution is identical with that given by (11.8) and (11.9) of section 11. The theoretical equations (17.5) and (17.6) do not involve either U

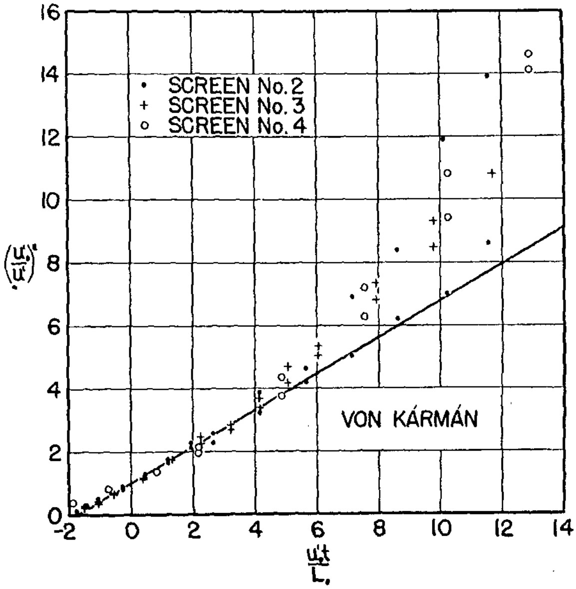


FIG. 8. Von Kármán's measurements of turbulence behind screens.

or M explicitly. However, for comparison with experimental data, they may be written as follows:

$$\left(\frac{u_0'}{u'}\right)^2 = 1 + \frac{2Au_0't}{L_0} = 1 + 2A\,\frac{u_0'}{U}\,\frac{(x - x_0)}{L_0} \tag{17.8}$$

$$\left(\frac{L}{L_0}\right)^2 = 1 + \frac{2Au_0't}{L_0} = 1 + 2A\,\frac{u_0'}{U}\,\frac{(x - x_0)}{L_0} \tag{17.9}$$

Both u_0' and L_0 should be known, but unfortunately L_0 was not measured in all of the experiments.

Figures 7, 8, and 9 show the results of Hall, of von Kármán, and of the author and his associates (designated NBS) plotted in a manner to facilitate comparison with equation (17.8).

The reference position x_0 has been taken as 40 times the mesh length except for von Kármán's results for which x_0 was taken as 212.5 times the rod diameter (equivalent to $x_0/M=40$ for $d/M=0.188$). In the absence of definite information as to L_0, L_0/M was assumed equal to 0.29 except for von Kármán's results for which L_0/d was assumed to be 1.54 (equivalent to $L_0/M=0.29$ for $d/M=0.188$). The value of u_0' was determined by interpolation from the observations of each experimenter near $x/M=40$, giving the following results.

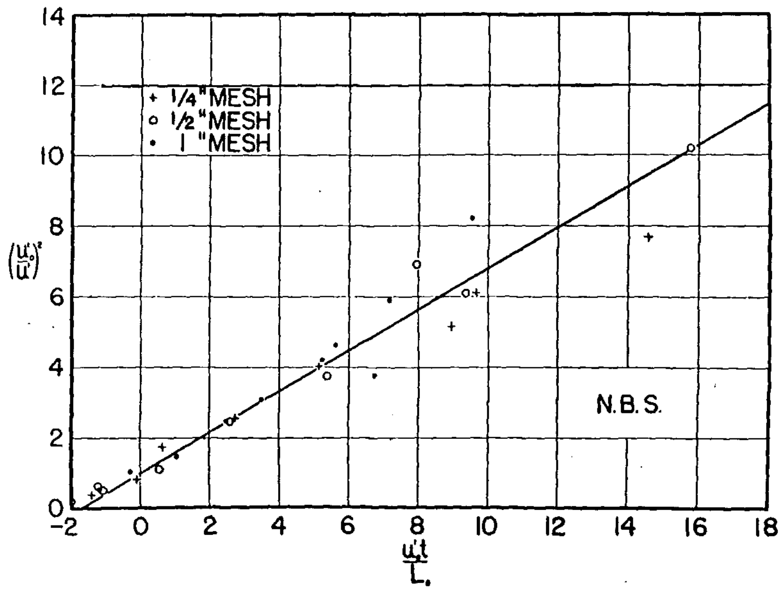


FIG. 9. NBS measurements of turbulence behind screens.

Experimenter	*Mesh Inches*	*Rod Diameter Inches*	*Air Speed ft/sec*	u_0'/U	*Remarks*
Hall	1.0	0.188	20	0.0146	
	0.5	.092	20	.0144	
	0.5	.092	40	.0152	
	0.5	.092	80	.0174	
von Kármán	0.5	0.105	38 and 54	.0201	Screen 2
	0.5	.084	38 and 75	.0201	Screen 3
	0.5	.043	38 and 75	.0299	Screen 4
NBS	0.25	0.050	20–70	0.0250	NACA Tech. Rept. 581
	0.5	.096	20–70	.0221	NACA Tech. Rept. 581
	1.0	.196	20–70	.0224	NACA Tech. Rept. 581
	1.0	0.196	30	0.0188	Recent tests
	1.0	.196	70	.0173	Recent tests

The values of u_0'/U range from 0.0144 to 0.0299; presumably the differences are due mainly to the factors discussed in section 11, although systematic errors may be partly responsible.

In each figure, equation (17.8) with constant A equal to 0.29 is plotted as a straight line. Most of the points would be better fitted by a curve of increasing slope with increasing time. It thus appears that equation (11.9) with $(A+B)/A$ having some value between 1 and 2 fits the experimental data better than (17.8).

However, the data are not at all consistent. The departures are largest for the smaller values of u'/U. In Hall's experiments, the results on the $\frac{1}{2}$-inch screen show little systematic departure at 40 and 80 ft/sec, whereas those on the same screen at 20 ft/sec and on the 1-inch screen begin to rise above the line at $u_0't/L_0=0.5$. Von Kármán's data on screen 2 at 38 ft/sec lie near the line; those on the same screen at 54 ft/sec and on screens 3 and 4 at 38 and 75 ft/sec begin to rise above the line at $u_0't/L_0=4.0$. The older results of the author and his associates, while scattered, agree with the line within 12 percent to $u_0't/L_0=18$; the more recent results begin to rise above the line at $u_0't/L_0=0.5$ and are in fair agreement with Hall's data on a 1-inch screen. Unfortunately, data at large values of x/M could not be obtained in the recent experiments.

Thus, even when attention is confined to the behavior of the isotropic turbulent field, there remain discrepancies in the experimental data such that no definite conclusions can be drawn as to the merits of any theory. Further experiments are required under carefully controlled conditions in an air stream of low turbulence over a wide range of values of x/M and with due regard to the various systematic errors that may be present. These experiments would be of the greatest value if the scale were also measured.

18. The length λ. Relation between λ and L. The general expression for the mean rate of dissipation in the flow of a viscous fluid is:

$$W = \mu\left\{2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} + \overline{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2} + \overline{\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2} + \overline{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2}\right\} \tag{18.1}$$

where μ is the viscosity.

For isotropic turbulence this becomes:

$$\frac{W}{\mu} = 6\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 6\overline{\left(\frac{\partial u}{\partial y}\right)^2} + 6\,\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} \tag{18.2}$$

which, from the relations given in section 14, reduces to:

$$W = -7.5\,\mu\,\overline{u^2}\,g_0'' = 7.5\,\mu\,\overline{(\partial u/\partial y)^2}. \tag{18.3}$$

But g_0'' is defined by:

$$g_0'' = -2 \lim_{r\to 0} \left(\frac{1-g}{r^2}\right) \tag{18.4}$$

and has the dimensions of the reciprocal of the square of a length. Let

$$g_0'' = -2/\lambda^2 \tag{18.5}$$

the factor 2 being introduced to conform to Taylor's definition of λ. Then

$$W = 15\mu\overline{u^2}/\lambda^2. \tag{18.6}$$

The length λ may be interpreted in several ways. Equation (18.6) may be considered a definition, λ being regarded roughly as a measure of the diameters of the smallest eddies which are responsible for the dissipation of energy. Or, since $1/\lambda^2 = \lim_{r\to 0}(1-g)/r^2 = \lim_{Y\to 0}(1-R_y)Y^2$, λ^2 is a measure of the radius of curvature of the R_y curve at $Y=0$. Or, if a parabola is drawn tangent to the R_y curve at $Y=0$, this parabola cuts the axis at the point $Y=\lambda$.

Since $W = -(3/2)\rho(d\overline{u^2}/dt)$, the decay law may be written:

$$d\overline{u^2}/dt = -10\nu\overline{u^2}/\lambda^2. \tag{18.7}$$

This result can also be derived directly from equation (16.5) as shown by von Kármán.

By comparing this expression for the decay law with that previously given (equation 17.2), namely

$$d\overline{u^2}/dt = -Au'^3/L \tag{18.8}$$

it is seen that

$$Au'/L = 10\nu/\lambda^2 \tag{18.9}$$

or, since $u'L/\nu = N_0$

$$\lambda^2/L^2 = 10/AN_0. \tag{18.10}$$

Introducing the experimental value of A,

$$\lambda/L = 6.97/\sqrt{N_0}. \tag{18.11}$$

A similar relation holds for λ/L_x where L_x is the longitudinal scale. If the Reynolds Number is formed from L_x, the numerical constant is approximately 4.93.

During the decay of self-preserving turbulence N_0 is constant and λ is proportional to L but the constant of proportionality varies inversely as $\sqrt{N_0}$ for turbulent fields of different Reynolds Number.

Although it cannot be expected on physical grounds that these relations hold at very low values of N_0, there is no experimental evidence of any departure from equations (18.7) and (18.8) for values of N_0 as low as 10. There seems to be no difficulty in drawing correlation curves for which λ is greater than L, but no such experimental curves have been measured. However, in an example quoted by Taylor,[15] λ/L is as great as 0.86.

19. The spectrum of turbulence, relation between spectrum and correlation. The description of turbulence in terms of intensity and scale resembles the description of the molecular motion of a gas by temperature and mean free path. A more detailed picture can be obtained by considering the distribution of energy among eddies of different sizes, or more conveniently the distribution of energy with frequency. Just as a beam of white light may be separated into a spectrum by the action of a prism or grating, the electric current produced by a hot wire anemometer subjected to the speed fluctuations may be analyzed by means of electric filters into a spectrum.

The mean value of $\overline{u^2}$ may be regarded as made up of a sum of contributions $\overline{u^2}F(n)dn$, where $F(n)$ is the contribution from frequencies between n and $n+dn$ and $\int_0^\infty F(n)dn = 1$. The curve of $F(n)$ plotted against n is the spectrum curve. According to the proof given by Rayleigh and quoted by Taylor[20]

$$F(n) = 2\pi^2 \lim_{T\to\infty} (I_1^2 + I_2^2)/T \tag{19.1}$$

where T is a long time and

$$\begin{aligned} I_1 &= (1/\pi) \int_0^T u \cos 2\pi nt \, dt \\ I_2 &= (1/\pi) \int_0^T u \sin 2\pi nt \, dt. \end{aligned} \tag{19.2}$$

When the fluctuations are superposed on a stream of mean velocity U and are very small in comparison with U, the changes in u at a fixed point may be regarded as due to the passage of a fixed turbulent pattern over the point, i.e., it may be assumed that

$$u = \phi(t) = \phi(x/U) \tag{19.3}$$

where x is measured upstream at time $t=0$ from the fixed point. The correlation R_x between the fluctuations at the times t and $t+x/U$ is defined by

$$R_x = \frac{\overline{\phi(t)\phi(t + x/U)}}{\overline{u^2}}. \tag{19.4}$$

It can be shown[20] that

$$\int_{-\infty}^{\infty} \phi(t)\phi(t + x/U)dt = 2\pi^2 \int_0^\infty (I_1^2 + I_2^2) \cos (2\pi nx/U)dn \tag{19.5}$$

[20] Taylor, G. I., *The spectrum of turbulence*, Proc. Roy. Soc. London, Ser. A, **164**, 476 (1938).

or, substituting for $I_1^2+I_2^2$ its value in terms of $F(n)$,

$$R_x = \int_0^\infty F(n)\cos(2\pi nx/U)dn \tag{19.6}$$

and

$$F(n) = (4/U)\int_0^\infty R_x \cos(2\pi nx/U)dx \tag{19.7}$$

In other words, the correlation coefficient R_x and $UF(n)/\sqrt{8\pi}$ are Fourier transforms. If either is measured, the other may be computed. R_x is the function denoted by f in section 13. The length λ, which was defined in terms of the function g or R_y, is related to R_x by the equation:

$$1/\lambda^2 = 2\lim_{x\to 0}(1-R_x)/x^2. \tag{19.8}$$

When n and x are small, $\cos(2\pi nx/U)$ in (19.6) may be approximated by $1-2\pi^2x^2n^2/U^2$. Hence

$$1/\lambda^2 = (4\pi^2/U^2)\int_0^\infty n^2F(n)dn. \tag{19.9}$$

If the turbulence is self-preserving, the shape of the correlation curve is a function of the Reynolds Number of the turbulence. Hence the spectrum curve is also a function of the Reynolds Number of the turbulence. Introducing the longitudinal scale L_x ($L_x=\int_0^\infty R_x dx$) in equation (19.9),

$$\frac{L_x^2}{\lambda^2} = 4\pi^2\int_0^\infty\left(\frac{nL_x}{U}\right)^2\frac{UF(n)}{L_x}\,d\left(\frac{nL_x}{U}\right) \tag{19.10}$$

and in equation (19.7),

$$\frac{UF(n)}{L_x} = 4\int_0^\infty R_x\cos\frac{2\pi nL_x}{U}\,\frac{x}{L_x}\,d\left(\frac{x}{L_x}\right) \tag{19.11}$$

both of which are expressed in terms of the non-dimensional variables $UF(x)/L_x$, nL_x/U, x/L_x, λ/L_x, and R_x. The mean speed U enters only in fixing the frequency scale.

Typical spectrum curves determined experimentally[21,22] are shown in Fig. 10. Studies of the relation between the spectrum and the correlation curve have been given by Taylor.[20]

From equation (19.10) it may be inferred that if the curve of $UF(n)/L_x$ vs.

---

[21] Simmons, L. F. G., and Salter, C., *An experimental determination of the spectrum of turbulence*, Proc. Roy. Soc. London Ser. A, **165**, 73 (1938).

[22] Dryden, H. L., *Turbulence investigations at the National Bureau of Standards*, Proc. Fifth Inter. Congr. Appl. Mech., Cambridge, Mass., 1938, p. 362.

nL_x/U is independent of U, L_x/λ should also be independent of U, which is contrary to the known dependence of L_x/λ on the Reynolds Number of the turbulence. Equation (19.9) shows that the value of λ is determined largely by the values of $F(n)$ at large values of n. The NPL measurements in Fig. 10

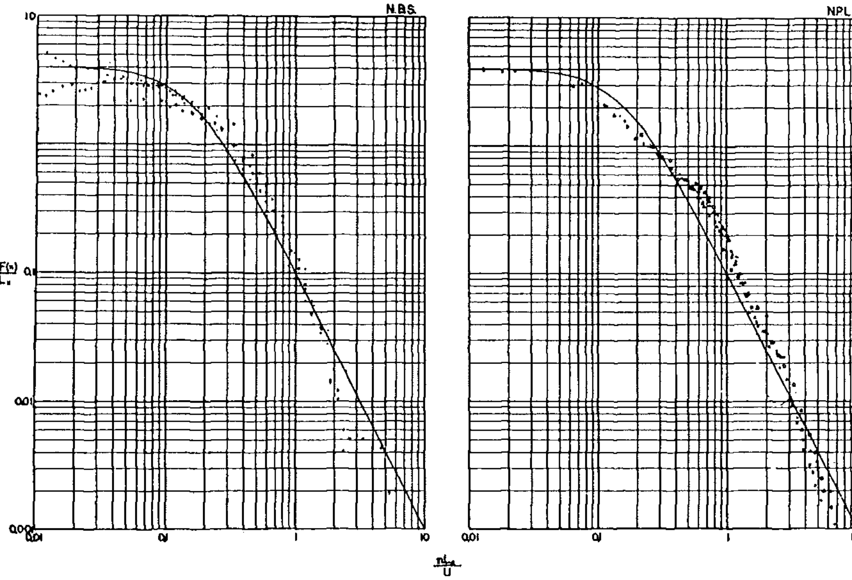


FIG. 10. Comparison of National Bureau of Standards and National Physical Laboratory measurements of the spectrum of turbulence, plotted non-dimensionally.

At left, NBS values 40(·) and 160(+) inches behind 1-inch mesh screen at 40 ft/sec.

At right, NPL values of ($F(n)$ from Table II of reference 21, L_x from reference 20) 82 inches behind 3-inch mesh screen at 15(·), 20(×), 25(+), 30(△), and 35(□) ft/sec.

The reference curve in each case is the curve

$$\frac{UF(n)}{L_x} = \frac{4}{1 + \frac{4\pi^2 n^2 L_x^2}{U^2}}$$

where U is the mean speed, L_x is the integral $\int_0^\infty R_x dx$, R_x is the correlation between the fluctuations at two points separated by the distance x in the direction of flow, n is the frequency, and $F(n)$ is the fraction of the total energy of the turbulence arising from frequencies between n and $n+dn$.

show clearly this dependence of the spectrum curve on U at high frequencies.

When the Reynolds Number of the turbulence is large, λ/L_x becomes small. Experimental measurements show that both R_x and R_y curves approach exponential curves. From integration of equation (13.9) it follows that $2L = L_x$ and equation (19.11) for the corresponding spectrum curve becomes:

$$\frac{UF(n)}{L_x} = \frac{4}{1 + 4\pi^2 n^2 L_x/U^2}\,. \tag{19.12}$$

This is the reference curve drawn in Fig. 10. As U decreases, λ increases, and the departures at large values of nL_x/U becomes greater. The changes in the total energy of the fluctuations associated with these changes in the spectrum at high frequencies are extremely small.

Adopting this expression for the spectrum curve, it is possible to compute the effect of varying the cut-off frequency of the measuring equipment on the measured value of the energy of the fluctuations. If the equipment passes

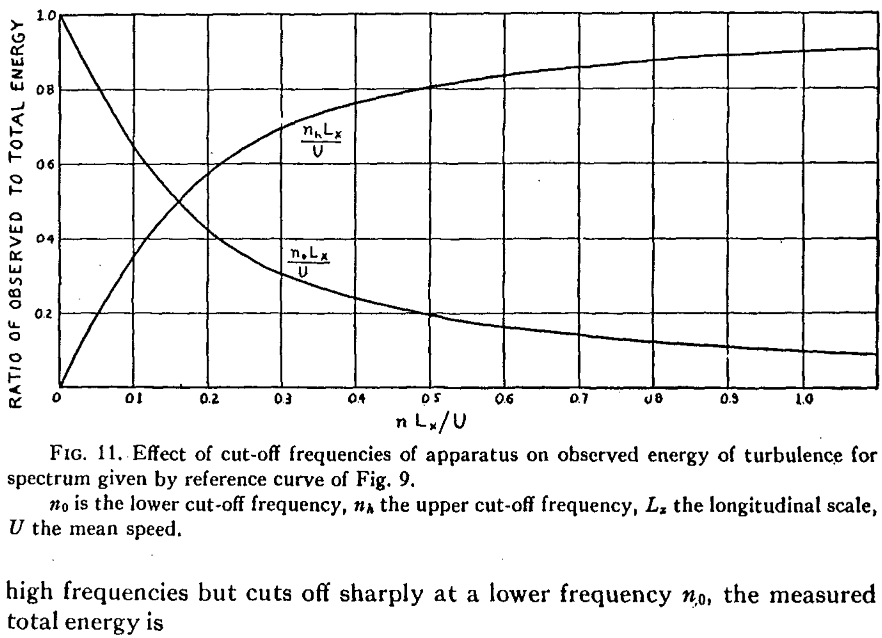


FIG. 11. Effect of cut-off frequencies of apparatus on observed energy of turbulence for spectrum given by reference curve of Fig. 9.

n_0 is the lower cut-off frequency, n_h the upper cut-off frequency, L_x the longitudinal scale, U the mean speed.

high frequencies but cuts off sharply at a lower frequency n_0, the measured total energy is

$$\tfrac{1}{2}\rho\overline{u^2}\int_{n_0L_x/U}^{\infty}\frac{4(L_x/U)dn}{1+4\pi^2n^2L_x^2/U^2}=\left(1-\frac{4}{2\pi}\tan^{-1}\frac{2\pi n_0L_x}{U}\right)\tfrac{1}{2}\rho\overline{u^2}. \qquad (19.13)$$

The ratio of the observed to the actual total energy is shown in Fig. 11 for various values of n_0L_x/U.

Similarly, if the equipment passes low frequencies but cuts off sharply at a higher frequency n_h, the measured total energy is $(4/2\pi)\tan^{-1} 2\pi n_hL_x/U(\tfrac{1}{2}\rho\overline{u^2})$. The ratio of the observed to the actual total energy for high frequency cut-off is also shown in Fig. 10.

The fact that the correlation and spectrum curves are of the exponential type has been interpreted[22] as meaning that turbulence is a generalized chance phenomenon, as nearly chance as a continuous curve can be and retain its continuity.

20. Fluctuating pressure gradients. In theories of the effect of turbulence on transition in boundary layers, it is desired to know the value of the root-mean-square pressure gradients, i.e., $\overline{(\partial p/\partial x)^2}$, $\overline{(\partial p/\partial y)^2}$, and $\overline{(\partial p/\partial z)^2}$. Taylor has shown[8] that

$$\sqrt{\overline{(\partial p/\partial x)^2}} = 2\sqrt{2}\,\rho\overline{u^2}/\lambda. \tag{20.1}$$

Combining this with the relation (18.11), i.e., $\lambda/L = 6.97/\sqrt{(\overline{u^2})^{1/2}L/\nu}$

$$\sqrt{\overline{\left(\frac{\partial p}{\partial x}\right)^2}} = \frac{2\sqrt{2}\cdot\rho\overline{u^2}}{6.97L}\sqrt{\frac{(\overline{u^2})^{1/2}L}{\nu}}\,. \tag{20.2}$$

The quantities $\sqrt{\overline{u^2}}$ and L occur in this expression in the combination $[(\sqrt{\overline{u^2}})/L^{1/5}]^{5/2}$. The ratio $(\sqrt{\overline{u^2}}/U)(D/L)^{1/5}$, where U is the mean speed and D the reference dimension of a body under study is known as the Taylor turbulence parameter.

21. The diffusive character of turbulence. An early experimental distinction between turbulent and non-turbulent flow was based on the observation that a filament of dye introduced into a turbulent fluid stream is rapidly diffused over the entire cross section of the stream whereas in a non-turbulent flow the filament retains its identity although it may show some waviness. It has been pointed out in section 7 that the effect of the turbulent fluctuations on the mean motion is the introduction of eddy stresses associated with the transfer of momentum by the diffusion of fluid particles. Von Kármán[23] has given a useful account of the mechanism of the diffusion of discrete particles and its effect in producing a shearing stress. A theory of diffusion by continuous movements has been developed by Taylor.[13] The process of diffusion has been found helpful in the experimental study of the statistical properties of turbulence.

22. Diffusion by continuous movements. Consider in a uniform isotropic turbulent field the displacement X and velocity u parallel to the arbitrarily selected x axis. The intensity $\sqrt{\overline{u^2}}$ is constant, the field being assumed uniform. Let u_t and $u_{t'}$ be the values of u at times t and t' respectively. Consider the definite integral $\int_0^t u_t u_{t'} dt'$. Introducing the correlation coefficient $R_{t,t'}$ between u_t and $u_{t'}$, remembering that $\overline{u^2}$ is constant,

$$\int_0^t \overline{u_t u_{t'}}\,dt' = \overline{u^2}\int_0^t R_{t,t'}\,dt'. \tag{22.1}$$

Let $t'-t=T$ and place $R_{t,t'}=R_T$. Since R_T is an even function of T, (22.1) may be written:

$$\int_0^t \overline{u_t u_{t'}}\,dt' = \overline{u^2}\int_0^t R_T\,dT. \tag{22.2}$$

[23] Kármán, Th. von, *Turbulence*, Jour. Roy. Aeron. Soc. **41**, 1109 (1937).

But

$$\int_0^t \overline{u_t u_{t'}} dt' = \overline{u_t \int_0^t u_{t'} dt'} = \overline{u_t X} = \overline{uX}. \tag{22.3}$$

Hence

$$\overline{u^2} \int_0^t R_T dT = \overline{uX} = (1/2) d\overline{X^2}/dt. \tag{22.4}$$

When the time t is so small that R_T approximates unity, equation (22.4) becomes:

$$(1/2) d\overline{X^2}/dt = \overline{u^2} t$$

or

$$\sqrt{\overline{X^2}} = \sqrt{\overline{u^2}}\, t. \tag{22.5}$$

If R_T is equal to zero for all times greater than some time T_0

$$\overline{uX} = \overline{u^2} \int_0^{T_0} R_T dT = \text{constant}. \tag{22.6}$$

Define a length l_1 by the relation:

$$l_1 = \sqrt{\overline{u^2}} \int_0^{T_0} R_T dT \tag{22.7}$$

whence

$$l_1 \sqrt{\overline{u^2}} = \overline{uX} = (1/2) d\overline{X^2}/dt \tag{22.8}$$

and

$$\overline{X^2} = 2 l_1 \sqrt{\overline{u^2}}\, t. \tag{22.9}$$

If $R_T = e^{-T/T_0}$, $l_1 = \sqrt{\overline{u^2}}\, T_0$ and the solution of (22.4) yields:

$$\overline{X^2} = 2\overline{u^2} T_0 \left[t - T_0 (1 - e^{-T/T_0}) \right]. \tag{22.10}$$

Equation (22.10) reduces to (22.5) when t is small compared to T_0 and to (22.9) when t is large compared to T_0.

The diffusion in a uniform field is accordingly completely determined by the correlation function R_T.

23. Diffusion in isotropic turbulence. The foregoing theory is directly applicable to diffusion in a uniform isotropic field. However no general statement can be made as to the relation between the length l_1 and the scale L defined in terms of the correlation coefficient R_y in section 9. For the turbulence behind a grid or honeycomb, Taylor found from an analysis of the available experimental results that L was approximately twice l_1.

The essential features of diffusion in isotropic turbulence expressed in equations (22.5) and (22.9) may be summarized as follows:

1. For time intervals which are small in comparison with the ratio of l_1 to $\sqrt{\overline{u^2}}$, the diffusing quantity spreads at a uniform rate proportional to the intensity $\sqrt{\overline{u^2}}$, and the rate is not dependent on the length l_1.

2. For time intervals which are large in comparison with the ratio of l_1 to $\sqrt{\overline{u^2}}$, the diffusing quantity N spreads in accordance with the usual diffusion equation

$$\frac{\partial N}{\partial t}+U\frac{\partial N}{\partial x}+V\frac{\partial N}{\partial y}+W\frac{\partial N}{\partial z}=\frac{\partial}{\partial x}\left(D\frac{\partial N}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial N}{\partial y}\right)+\frac{\partial}{\partial z}\left(D\frac{\partial N}{\partial z}\right)$$

with a coefficient of diffusion D equal to $l_1\sqrt{\overline{u^2}}$, where l_1 is a length defined by $\int_0^\infty R_T dT$.

3. For intermediate time intervals, the diffusion is dependent on the function R_T which represents the correlation between the speed of a particle at any instant and the speed of the same particle after a time interval T.

Consider the diffusion of heat from a hot wire placed in a uniform field of isotropic turbulence in a fluid stream of mean speed U. Observations of the lateral spread of the thermal wake at a distant x downstream may be used to compute the root-mean-square lateral displacement $\sqrt{\overline{Y^2}}$ of the heated particles during a time interval $t=x/U$.

It is convenient to characterize the spread by the angle subtended at the source by the two positions where the temperature rise is half that at the center of the wake. There is a lateral spread of heat produced by the ordinary molecular conduction corresponding to an angle α_0 in degrees of $190.8\sqrt{k/\rho c U x}$ where k, ρ, and c are thermal conductivity, density, and specific heat (at constant pressure) of the fluid. It may be shown that the total subtended angle α is related to the angle α_t produced by turbulent diffusion and α_0, as follows:

$$\alpha^2 = \alpha_t^2 + \alpha_0^2. \tag{23.1}$$

The temperature distribution in the wake follows an "error" curve as does the amplitude of the turbulent velocity fluctuations, so that the lateral displacement Y also has the same Gaussian frequency distribution. The value of the lateral spread at which the ordinate is half the maximum is $2.354\sqrt{\overline{Y^2}}$ for this distribution. Hence, expressing α_t in degrees,

$$\alpha_t = 134.7\sqrt{\overline{Y^2}}/x \tag{23.2}$$

whence from (22.5) for small values of x,

$$\alpha_t = 134.7\sqrt{\overline{v^2}}/U \tag{23.3}$$

where v is written in place of u in (22.5) since the diffusion in the v direction is being studied.

Thus an experiment on thermal diffusion provides a method of measuring $\sqrt{\overline{v^2}}$. The method was used by Schubauer[24] who showed that α_t was independent of speed over the range 10 to 50 ft/sec and also independent of x over the range 1/2 to 6 inches.

From measurements at large values of x, it is theoretically possible to compute the correlation curve, R_T, vs. T. In any actual experiment, however, the intensity of the turbulence will decrease with x to an extent that must be considered. As discussed by Taylor,[8] R_T may then be considered a function of $\eta = \int_0^T \sqrt{\overline{v^2}}dT = \int_0^x (\sqrt{\overline{v^2}}/U)dx$. The equation analogous to (22.4) for v and Y becomes:

$$(1/2)(U/\sqrt{\overline{v^2}})(d\overline{Y^2}/dx) = \int_0^{\eta_x} R_\eta d\eta. \tag{23.4}$$

The correlation is given by the expression

$$R_\eta = \frac{d}{d\eta}\left(\frac{U}{2\sqrt{\overline{v^2}}}\frac{d\overline{Y^2}}{dx}\right) \tag{23.5}$$

and thus involves a double differentiation of experimental curves, a process which is usually not very accurate.

24. Statistical theory of non-isotropic turbulence. In non-isotropic turbulence the description of the state of the turbulence becomes much more complex. The eddy shearing stresses do not vanish and the eddy normal stresses are not necessarily equal. Six quantities instead of one are required to specify the intensity. Similarly the correlation tensor cannot be expressed in terms of a single scalar function. In general six scalar functions are required. No theoretical investigation using these twelve functions has yet been carried out.

The exploration of this field is still in its earliest stages. Von Kármán[9,16] has given some discussion of energy transport and dissipation and vorticity transport, neglecting the triple correlations, and he has also presented a more detailed discussion of two-dimensional flow with constant shearing stress (Couette's problem). The advance of the theory is definitely handicapped by the absence of reliable experimental data on the twelve functions required to describe the state of turbulence.

25. Diffusion in non-isotropic turbulence. The only theoretical approach at present available for estimating the diffusion in non-isotropic turbulence is to consider the process as approximately equivalent to diffusion in isotropic turbulence of intensity equal to $\sqrt{\overline{v^2}}$ and scale l', where v is the component in the direction in which the diffusion is studied and l' is the length defined by an equation analogous to (22.7), namely,

---

[24] Schubauer, G. B., *A turbulence indicator utilizing the diffusion of heat*, Tech. Rept. Nat. Adv. Comm. Aeron., No. 524 (1935).

$$l' = \sqrt{\overline{v^2}} \int_0^\infty R_T dt. \tag{25.1}$$

In most experiments the length l' is not measured. Prandtl defined a mixing length l in terms of the shearing stress τ by the relation:

$$\tau = \rho l^2 \left| \frac{dU}{dy} \right| \frac{dU}{dy} . \tag{25.2}$$

This relation may be interpreted as an equation governing the diffusion of momentum with a coefficient of diffusion equal to $l^2 | dU/dy |$. Prandtl in fact assumed $\sqrt{\overline{v^2}}$ proportional to $l | dU/dy |$ and incorporated the factors of proportionality in the length l. It is obvious that

$$l^2 | dU/dy | = l' \sqrt{\overline{v^2}}. \tag{25.3}$$

The length l can be obtained experimentally if the distributions of velocity and shearing stress are known, and, if $\sqrt{\overline{v^2}}$ is also measured, l' may be computed. Sherwood and Woertz[25] have made an experimental study of these relationships.

Taylor[26] pointed out that fluctuating pressure gradients influence the transfer of momentum and suggested that the vorticity be taken as the property undergoing diffusion. The result was the well known vortex transport theory.

Both theories imply diffusion for a time interval long compared to $l' \sqrt{\overline{v^2}}$. When diffusion is studied near the source, experiment shows[27] a behaviour like that discussed in section 23. The spread is nearly linear with x, although unsymmetical in this case. It is probable that the unsymmetrical character cannot be explained on the basis of a single scalar diffusion coefficient.

26. Correlation in turbulent flow through a pipe. Taylor[28] has shown that the correlation between the component of velocity at a fixed point and that at a variable point in the same cross section must be negative for some positions of the variable point, if the applied pressure difference between the ends of the pipe is constant and the fluid may be considered incompressible. Suppose the mean velocity is U and the correlation R has been measured between

---

[25] Sherwood, T. K., and Woertz, B. B., *Mass transfer between phases, role of eddy diffusion*, Ind. Eng. Chem. **31**, 1034 (1939).

[26] Taylor, G. I., *Transport of vorticity and heat through fluids in turbulent motion*, Proc. Roy. Soc. London Ser. A, **135**, 685 (1932).

[27] Skramstad, H. K., and Schubauer, G. B., *The application of thermal diffusion to the study of turbulent air flow*, Phys. Rev., **53**, 927 (1938). Abstract only. Full paper not published. A few additional details are given in Dryden, Hugh L., *Turbulence and diffusion*, Ind. Eng. Chem. **31**, 416 (1939).

[28] Taylor, G. I., *Correlation measurements in a turbulent flow through a pipe*, Proc. Roy. Soc. London Ser. A, **157**, 537 (1936).

the component u_1 of the fluctuations at a fixed point P and u_2 at a variable point Q in the same cross section. Since the mean flow is constant,

$$\int (U + u_2)dydz = \int Udydz = \text{constant} \tag{26.1}$$

where the integration is taken over the cross section. At any instant,

$$\int u_2 dydz = 0. \tag{26.2}$$

Multiplying by u_1, which is constant for this integration and may be placed under the integral sign,

$$\int u_1 u_2 dydz = 0. \tag{26.3}$$

Since (26.3) is true for any instant, it is true for the integral over a time interval T. Hence

$$(1/T)\int_0^T \left[\int u_1 u_2 dydz\right] dt = 0. \tag{26.4}$$

Changing the order of the integration and remembering that $(1/T)\int_0^T u_1 u_2 dt = \overline{u_1 u_2}$

$$\int \overline{u_1 u_2} dydz = 0. \tag{26.5}$$

Introducing the correlation R,

$$\int R u_1' u_2' dydz = 0. \tag{26.6}$$

But u_1' is constant with respect to the integration and accordingly

$$\int R u_2' dydz = 0. \tag{26.7}$$

Since u_2' is positive, R must be negative for some positions of Q.

For a circular pipe (26.7) becomes:

$$\int_0^a u_r' R r dr = 0 \tag{26.8}$$

where u_r' is the value of $\sqrt{\overline{u^2}}$ at radius r and a is the radius of the pipe.

This relation was experimentally verified in experiments made by Simmons with the fixed point at the center of the pipe.

THE LOCAL STRUCTURE OF TURBULENCE IN INCOMPRESSIBLE VISCOUS FLUID FOR VERY LARGE REYNOLDS' NUMBERS

By A. KOLMOGOROFF, Member of the Academy

§ 1. We shall denote by

$$u_\alpha(P) = u_\alpha(x_1, x_2, x_3, t), \qquad \alpha = 1, 2, 3,$$

the components of velocity at the moment t at the point with rectangular cartesian co-ordinates x_1, x_2, x_3. In considering the turbulence it is natural to assume the components of the velocity $u_\alpha(P)$ at every point $P = (x_1, x_2, x_3, t)$ of the considered domain G of the four-dimensional space (x_1, x_2, x_3, t) random variables in the sense of the theory of probabilities (cf. for this approach to the problem Millionshtchikov ([1])).

Denoting by $\overline{A}$ the mathematical expectation of the random variable A we suppose that

$$\overline{u_\alpha^2} \quad \text{and} \quad \overline{\left(\frac{du_\alpha}{dx_\beta}\right)^2}$$

are finite and bounded in every bounded subdomain of the domain G.

Introduce in the four-dimensional space (x_1, x_2, x_3, t) new co-ordinates

$$\left.\begin{aligned} y_\alpha &= x_\alpha - x_\alpha^{(0)} - u_\alpha(P^{(0)})(t - t^{(0)}) \\ s &= t - t^{(0)} \end{aligned}\right\} \qquad (1)$$

where

$$P^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t)$$

is a certain fixed point from the domain G. Observe that the co-ordinates y_α of any point P depend on the random variables $u_\alpha(P^{(0)})$ and hence are themselves random variables. The velocity components in new co-ordinates are

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)}). \qquad (2)$$

Suppose that for some fixed values of $u_\alpha(P^{(0)})$ the points $P^{(k)}$, $k = 1, 2, \ldots, n$, having in the co-ordinate system (1) the co-ordinates $y_\alpha^{(k)}$ and $s^{(k)}$, are situated in the domain G. Then we may define a $3n$-dimensional distribution law of probabilities F_n for the quantities

$$w_\alpha^{(k)} = w_\alpha(P^{(k)}), \qquad \alpha = 1, 2, 3; \quad k = 1, 2, \ldots, n,$$

2 C. R. (Doklady) de l'Acad. des Sci. de l'URSS, 1941, v. XXX, № 4.

where

$$u_\alpha^{(0)} = u_\alpha(P^{(0)})$$

are given.

Generally speaking, the distribution law F_n depends on the parameters $x_\alpha^{(0)}$, $t^{(0)}$, $u_\alpha^{(0)}$, $y_\alpha^{(k)}$, $s^{(k)}$.

Definition 1. *The turbulence is called locally homogeneous in the domain G, if for every fixed n,* $y_\alpha^{(k)}$ *and* $s^{(k)}$ *the distribution law* F_n *is independent from* $x_\alpha^{(0)}$, $t^{(0)}$ *and* $u_\alpha^{(0)}$ *as long as all points* $P^{(k)}$ *are situated in G.*

Definition 2. *The turbulence is called locally isotropic in the domain G, if it is homogeneous and if, besides, the distribution laws mentioned in definition 1 are invariant with respect to rotations and reflections of the original system of co-ordinate axes* (x_1, x_2, x_3).

In comparison with the notion of *isotropic turbulence* introduced by Taylor (²) our definition of locally isotropic turbulence is *narrower* in the sense that in our definition we demand the independence of the distribution law F_n from $t^{(0)}$, i. e. steadiness in time, and is *wider* in the sense that restrictions are imposed only on the distribution laws of differences of velocities and not of the velocities themselves.

§ 2. The hypothesis of isotropy in the sense of Taylor is experimentally quite well confirmed in the case of turbulence caused by passing of a flow through a grid (cf. (3)). In the majority of other cases interesting from the practical point of view it may by considered only as a rather far approximation of reality even for small domains G and very large Reynolds' numbers.

On the other hand we think it rather likely that in an arbitrary turbulent flow with a sufficiently large Reynolds' number *

$$R = \frac{LU}{\nu}$$

the hypothesis of local isotropy is realized with good approximation in sufficiently small domains G of the four-dimensional space (x_1, x_2, x_3, t) not lying near the boundary of the flow or its other singularities. By a «small domain» we mean here a domain, whose linear dimensions are small in comparison with L and time dimensions — in comparison with

$$T = \frac{U}{L}.$$

It is natural that in so general and somewhat indefinite a formulation the just advanced proposition cannot be rigorously proved **. In

* Here L and U denote the typical length and velocity for the flow in the whole.

** We may indicate here only certain general considerations speaking for the advanced hypothesis. For very large R the turbulent flow may be thought of in the following way: on the averaged flow (characterized by the mathematical expectations $\bar{u}_\alpha$) are superposed the «pulsations of the first order» consisting in disorderly displacements of separate fluid volumes, one with respect to another, of diameters of the order of magnitude $l^{(1)} = l$ (where l is the Prandtl's mixing path); the order of magnitude of velocities of these relative velocities we denote by $v^{(1)}$. The pulsations of the first order are for very large R in their turn unsteady, and on them are superposed the pulsations of the second order with mixing path $l^{(2)} < l^{(1)}$ and relative velocities $v^{(2)} < v^{(1)}$; such a process of successive refinement of turbulent pulsations may be carried until for the pulsations of some sufficiently large order n the Reynolds' number

$$R^{(n)} = \frac{l^{(n)} v^{(n)}}{\nu}$$

order to make its experimental verification possible in particular cases we indicate here a number of consequences of the hypothesis of local isotropy.

§ 3. Denoting by y the vector with components y_1, y_2, y_3, we consider the random variables

$$w_\alpha(y) = w_\alpha(y_1, y_2, y_3) = u_\alpha(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3,\ t) - u_\alpha(x_1, x_2, x_3, t). \quad (3)$$

In virtue of the assumed local isotropy their distribution laws are independent from x_1, x_2, x_3 and t. For the first moments of the quantities $w_\alpha(y)$ from the local isotropy follows that

$$\overline{w_\alpha(y)} = 0. \quad (4)$$

We proceed therefore to the consideration of the second moments *

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \overline{w_\alpha(y^{(1)})\, w_\beta(y^{(2)})}. \quad (5)$$

From the local isotropy follows that

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \frac{1}{2}\Big[B_{\alpha\beta}(y^{(1)}, y^{(1)}) + B_{\alpha\beta}(y^{(2)}, y^{(2)}) - - B_{\alpha\beta}(y^{(2)} - y^{(1)}, y^{(2)} - y^{(1)}) \Big]. \quad (6)$$

In virtue of this formula we may confine ourselves to the second moments of the form $B_{\alpha\beta}(y, y)$. For them

$$B_{\alpha\beta}(y, y) = \overline{B}(r) \cos\theta_\alpha \cos\theta_\beta + \delta_{\alpha\beta} B_{nn}(r), \quad (7)$$

where

$$r^2 = y_1^2 + y_2^2 + y_3^2, \quad y_\alpha = r\cos\theta_\alpha, \quad \delta_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta, \quad \delta_{\alpha\beta} = 1 \text{ for } \alpha = \beta$$

$$\overline{B}(r) = B_{dd}(r) - B_{nn}(r), \quad (8)$$

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[w_1(r, 0, 0)]^2}, \\ B_{nn}(r) &= \overline{[w_2(r, 0, 0)]^2}. \end{aligned}\right\} \quad (9)$$

---

becomes so small that the effect of viscosity on the pulsations of the order n finally prevents the formation of pulsations of the order $n + 1$.

From the energetical point of view it is natural to imagine the process of turbulent mixing in the following way: the pulsations of the first order absorb the energy of the motion and pass it over successively to pulsations of higher orders. The energy of the finest pulsations is dispersed in the energy of heat due to viscosity.

In virtue of the chaotical mechanism of the translation of motion from the pulsations of lower orders to the pulsations of higher orders, it is natural to assume that in domains of the space, whose dimensions are small in comparison with $l^{(1)}$, the fine pulsations of the higher orders are subjected to approximately space-isotropic statistical regime. Within small time-intervals it is natural to consider this regime approximately steady even in the case, when the flow in the whole is not steady.

Since for very large R the differences

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)})$$

of the velocity components in neighbouring points P and $P^{(0)}$ of the four-dimensional space (x_1, x_2, x_3, t) are determined nearly exclusively by pulsations of higher orders, the scheme just exposed leads us to the hypothesis of local isotropy in small domains G in the sense of definitions 1 and 2.

* All results of § 3 are quite similar to that obtained in (1), (2) and (4) for the case of isotropic turbulence in the sense of Taylor.

For $r=0$ we have

$$B_{dd}(0)=B_{nn}(0)=\frac{\partial}{\partial r}B_{dd}(0)=\frac{\partial}{\partial r}B_{nn}(0)=0, \tag{10}$$

$$\left.\begin{aligned}\frac{\partial^2}{\partial r^2}B_{dd}(0)=2\left(\frac{\partial w_1}{\partial y_1}\right)^2=2a^2,\\ \frac{\partial^2}{\partial r^2}B_{nn}(0)=2\left(\frac{\partial w_2}{\partial y_1}\right)^2=2a_n^2.\end{aligned}\right\} \tag{11}$$

The formulae (6)—(11) were obtained without use of the assumption of incompressibility of the fluid. From this assumption follows the equation

$$r\frac{\partial B_{dd}}{\partial r}=-2\overline{B}, \tag{12}$$

enabling us to express B_{nn} through B_{dd}. From (12) and (11) follows that

$$a_n^2=2a^2. \tag{13}$$

It is, further, easy to calculate that (assuming the incompressibility) the average dispersion of energy in unit of time per unit of mass is equal to

$$\bar{\varepsilon}=\nu\left\{2\overline{\left(\frac{\partial w_1}{\partial y_1}\right)^2}+2\overline{\left(\frac{\partial w_2}{\partial y_2}\right)^2}+2\overline{\left(\frac{\partial w_3}{\partial y_3}\right)^2}+\overline{\left(\frac{\partial w_2}{\partial y_1}+\frac{\partial w_1}{\partial y_2}\right)^2}+\right.$$
$$\left.+\overline{\left(\frac{\partial w_3}{\partial y_2}+\frac{\partial w_2}{\partial y_3}\right)^2}+\overline{\left(\frac{\partial w_1}{\partial y_3}+\frac{\partial w_3}{\partial y_1}\right)^2}\right\}=15\nu a^2. \tag{14}$$

§ 4. Consider the transformation of co-ordinates

$$y'_\alpha=\frac{y_\alpha}{\eta}, \qquad s'=\frac{s}{\sigma}. \tag{15}$$

The velocities, the kinematical viscosity and the average dispersion of energy are expressed in the new system of co-ordinates by the following formulae:

$$w'_\alpha=w_\alpha\frac{\sigma}{\eta}, \quad \nu'=\nu\frac{\sigma}{\eta^2}, \quad \bar{\varepsilon}'=\bar{\varepsilon}\frac{\sigma^3}{\eta^2}. \tag{16}$$

We introduce now the following hypothesis:

The first hypothesis of similarity. *For the locally isotropic turbulence the distributions F_n are uniquely determined by the quantities ν and $\bar{\varepsilon}$.*

The transformation of co-ordinates (15) leads for

$$\eta=\lambda=\sqrt{\frac{\nu}{a}}=\frac{\nu^{3/4}}{\bar{\varepsilon}^{1/4}} \tag{17}$$

and

$$\sigma=\frac{1}{a}=\sqrt{\frac{\nu}{\bar{\varepsilon}}} \tag{18}$$

to the quantitites $\nu'=1$, $\bar{\varepsilon}'=1$.

In virtue of the accepted hypothesis of similarity the corresponding function

$$B'_{dd}(r')=\beta_{dd}(r') \tag{19}$$

must be the same for all cases of locally isotropic turbulence. The formula

$$B_{dd}(r)=\sqrt{\nu\bar{\varepsilon}}\,\beta_{dd}\left(\frac{r}{\lambda}\right) \tag{20}$$

shows in combination with the already deduced that in the case of locally isotropic turbulence the second moments $B_{\epsilon\beta}(y^{(1)}, y^{(2)})$ are uniquely expressed through ν, $\bar{\varepsilon}$ and the universal function β_{dd}.

§ 5. In order to determine the behaviour of the function $\beta_{dd}(r')$ for large r' we introduce another hypothesis:

The second hypothesis of similarity*. *If the moduli of the vectors $y^{(k)}$ and of their differences $y^{(k)} - y^{(k')}$ (where $k \neq k'$) are large in comparison with λ, then the distribution laws F_n are uniquely determined by the quantity $\bar{\varepsilon}$ and do not depend on ν.*

Put

$$y''_\alpha = \frac{y'_\alpha}{k^3}, \quad s'' = \frac{s'}{k^2}; \tag{21}$$

where y'_α and s' are determined in accordance with the formulae (15), (17) and (18). Since for every k $\bar{\varepsilon}' = \bar{\varepsilon}'' = 1$, for r' large in comparison with $\lambda' = 1$ we have in virtue of the accepted hypothesis

$$B''_{dd}(r'') \backsim B'_{dd}(r'') = \beta_{dd}\left(\frac{r'}{k^3}\right).$$

On the other hand, from the formula (20) follows that

$$B''_{dd}(r'') = \frac{1}{k^2} B'_{dd}(r') = \frac{1}{k^2}\beta_{dd}(r').$$

Thus for large r'

$$\beta_{dd}\left(\frac{r'}{k^3}\right) \backsim \frac{1}{k^2}\beta_{dd}(r'),$$

whence

$$\beta_{dd}(r') \backsim C(r')^{2/3}, \tag{22}$$

where C is an absolute constant. In virtue of (17), (20) and (22) we have for r large in comparison with λ

$$B_{dd}(r) \backsim C\bar{\varepsilon}^{2/3} r^{2/3}. \tag{23}$$

From (23) and (12) it is easy to deduce that for r large in comparison with λ

$$B_{nn}(r) \backsim \frac{4}{3} B_{dd}(r). \tag{24}$$

As regards the last formula, observe that for r small in comparison with λ in virtue of (13) holds the relation

$$B_{nn}(r) \backsim 2B_{dd}(r). \tag{25}$$

Received
28. XII. 1940.

REFERENCES

[1] Millionshtchikov, C. R. Acad. Sci. URSS, XXII, 236 (1939). [2] Taylor, Proc. Roy. Soc. A, 151, 429—454 (1935). [3] Modern developments in fluid dynamics, edited by S. Goldstein (Oxford), vol. I, § 95 (1938). [4] Kármán, Journ. Aeronaut. Sci., 4, 131—138 (1937).

Translated by V. Levin.

---

* In terms of the schematical representation of turbulence developed in the footnote **, λ is the scale of the finest pulsations, whose energy is directly dispersed into heat energy due to viscosity. The sense of the second hypothesis of similarity consists in that the mechanism of translation of energy from larger pulsations to the finer ones is for pulsations of intermediate orders, for which $l^{(k)}$ is large in comparison with λ, independent from viscosity.

ON DEGENERATION OF ISOTROPIC TURBULENCE IN AN INCOMPRESSIBLE VISCOUS LIQUID

By A. N. KOLMOGOROFF, Member of the Academy

Similarly as in([1]) and([2]) we shall consider the components

$$u_\alpha(P,\ t)=u_\alpha(x_1,\ x_2,\ x_3,\ t)$$

of the velocity at the moment t at the point $P=(x_1,\ x_2,\ x_3)$ as random variables, and denote by $\overline{A}$ the mathematical expectation of the random variable A.

In the case of isotropic turbulence in the sense of Taylor ([3],[4])

$$\overline{u}_\alpha=0 \tag{1}$$

and the second moments

$$b_{\alpha\beta}=\overline{u_\alpha(x_1^{(1)},x_2^{(1)},x_3^{(1)},t)\,u_\beta(x_1^{(2)},x_2^{(2)},x_3^{(2)},t)} \tag{2}$$

are expressed by formulae ([1],[5])

$$b_{\alpha\beta}=\overline{b}\,(r,t)\cos\theta_\alpha\cos\theta_\beta+\delta_{\alpha\beta}\,b_{nn}(r,t) \tag{3}$$

where

$$r^2=(x_1^{(2)}-x_1^{(1)})^2+(x_2^{(2)}-x_2^{(1)})^2+(x_3^{(2)}-x_3^{(1)})^2,$$

$$x_\alpha^{(2)}-x_\alpha^{(1)}=r\cos\theta_\alpha,\ \ \delta_{\alpha\beta}=0\ \text{for}\ \alpha\neq\beta,\ \ \delta_{\alpha\beta}=1\ \text{for}\ \alpha=\beta$$

$$\overline{b}\,(r,t)=b_{dd}(r,t)-b_{nn}(r,t) \tag{4}$$

$$b_{dd}(r,t)=\overline{u_1(x_1,x_2,x_3,t)\,u_1(x_1+r,x_2,x_3,t)} \tag{5}$$

$$b_{nn}(r,t)=\overline{u_2(x_1,x_2,x_3,t)\,u_2(x_1+r,x_2\,x_3,t)} \tag{6}$$

$$\frac{\partial b_{dd}}{\partial r}=-\frac{2}{r}\,\overline{b}\,. \tag{7}$$

The formula (7) together with

$$b_{dd}(0,t)=b_{nn}(0,t)=\overline{[u_\alpha(x_1,x_2,x_3,t)]^2}=b\,(t) \tag{8}$$

enables us to determine the b_{nn} through b_{dd}. Thus, all second moments are determined by one function only, namely $b_{dd}(r,t)$.

For $b_{dd}(r,t)$ Kárman and Howarth([6]) obtained the equation

$$\frac{\partial b_{dd}}{\partial t}+2\left(\frac{\partial b_{nnd}}{\partial r}+\frac{4}{r}b_{nnd}\right)=2\nu\left(\frac{\partial^2 b_{dd}}{\partial r^2}+\frac{4}{r}\,\frac{\partial b_{dd}}{\partial r}\right) \tag{9}$$

where

$$b_{nnd}(r,t)=\overline{u_2^2(x_1,x_2,x_3,t)u_1(x_1+r,x_2,x_3,t)}\,. \tag{10}$$

Loitsiansky([5]) deduced from the equation (9) that

$$\Lambda = \int_0^\infty b_{nn}(r, t)\, r^4 dt \tag{11}$$

does not depend on time. Take for the «scale of turbulence» the length

$$L = \left(\frac{\Lambda}{b}\right)^{\frac{1}{5}}. \tag{12}$$

By L and the mean value of the velocity components

$$\mathbf{u} = \sqrt{b} \tag{13}$$

we determine the Reynolds number

$$R = \frac{L\mathbf{u}}{\nu} = \frac{\Lambda}{\nu}\mathbf{u}^{3/5}. \tag{14}$$

Since $\mathbf{u} \to 0$ for $t \to +\infty$, from (14) follows that for large t the Reynolds number R is small. At this final stage of degeneration of isotropic turbulence it is legal to apply the theory, neglecting the third moments. As has been shown in([1]), in this case

$$b_{dd}(r, t) = \frac{k}{(4\nu t)^{5/2}} \exp\left(-\frac{r^2}{8\nu t}\right) \tag{15}$$

$$b_{nn}(r, t) = \frac{k}{(4\nu t)^{5/2}} \left(1 - \frac{r^2}{8\nu t}\right) \exp\left(-\frac{r^2}{8\nu t}\right) \tag{16}$$

and, consequently,

$$\mathbf{u} = At^{-5/4} \tag{17}$$

$$L = B\sqrt{t}. \tag{18}$$

It is easy to see, how k, A and B are expressed through ν and Λ.

For large R Kármán([6]) has found, assuming $b_{dd}(r, t)$ to have the form

$$b_{dd}(r, t) = b\psi\left(\frac{r}{L}\right), \tag{19}$$

that

$$\mathbf{u} = \mathbf{u}_0 (t - t_0)^{-p}, \qquad L = L_0 (t - t_0)^{1-p}, \tag{20}$$

but did not, however, determine p*.

Below we give a new proof of the formulae (20), from which follows that

$$p = \frac{5}{7}. \tag{21}$$

In the first place we establish that for sufficiently large R from the assumption (19) follows

$$\frac{db}{dt} = -Kb^{3/2}L^{-1} \tag{22}$$

where K is an absolute constant. The formula (22) may be established in different ways. If we take the assumptions of my note([2]), we may argue as follows. For isotropic turbulence the mean energy per unit of

* The equation (77) in ([6]) is wrongly integrated. In fact, the exponent in the formula (78) should be equal to $\frac{AB}{5 + AB}$, and in the formula (81)—to $\frac{-5}{5 + AB}$. The value $p = \frac{5}{7}$ corresponds to $AB = 2$.

mass is equal to $\frac{3}{2}b$. Therefore the dispersion of energy in the unit of time per unit of mass is in the mean equal to

$$\bar{\varepsilon} = -\frac{3}{2}\frac{db}{dt}. \tag{23}$$

Since, using the notation $B_{dd}(r)$ from([2]), we have

$$b_{dd}(r) = b - \frac{1}{2}B_{dd}(r), \tag{24}$$

in virtue of the formula (23) from([2]) we have for r, small in comparison to L, but large in comparison to λ,

$$b_{dd}(r) \sim b\left(1 - \frac{C}{2b}\bar{\varepsilon}^{2/3}r^{2/3}\right). \tag{25}$$

For the function $\psi(\rho)$ we obtain from (25) for small ρ*

$$\psi(\rho) \sim 1 - g\rho^{2/3} \tag{26}$$

where

$$g = \frac{C}{2}\bar{\varepsilon}^{2/3}L^{2/3}b^{-1}.$$

Consequently

$$\bar{\varepsilon} = \left(\frac{2gb}{C}\right)^{3/2}L^{-1}. \tag{27}$$

Putting

$$K = \frac{2}{3}\left(\frac{2g}{C}\right)^{3/2} \tag{28}$$

we obtain from (23) and (27) the formula (22). The formula (28) shows that the coefficient K is expressed through the absolute constant C, introduced in([2]), and the quantity g, which is to be determined from (26) by the form of the function $\psi(\rho)$. From (12) and (10) we obtain

$$\frac{db}{dt} = -K\Lambda^{-\frac{1}{5}}b^{\frac{17}{10}}. \tag{29}$$

The integration of the equation (29) leads to the result

$$b = \left(\frac{10}{7K}\right)^{\frac{10}{7}}\Lambda^{\frac{2}{7}}(t - t_0)^{-\frac{10}{7}}. \tag{30}$$

From (30), (13) and (12) follows

$$u = \left(\frac{10}{7K}\right)^{\frac{5}{7}}\Lambda(t - t_0)^{-\frac{5}{7}} \tag{31}$$

$$L = \left(\frac{7K}{10}\right)^{\frac{2}{7}}\Lambda^{\frac{1}{7}}(t - t_0)^{\frac{2}{7}}. \tag{32}$$

Comparing (31) and (32) with (20), we find that $p = \frac{5}{7}$ indeed.

Institute of Mathematics.
Academy of Sciences of the USSR.
Moscow.

Received
4. III. 1941.

REFERENCES

[1] Millionshtchikov, C. R. Acad. Sci. URSS, XXII, 231 (1939). [2] Kolmogoroff, ibid., XXX (1941). [3] Taylor, Proc. Roy. Soc., A, **151**, 429 (1935). [4] S. Goldstein, Modern developments in fluid dynamics, I, § 91, Oxford (1938). [5] Лойцянский, Тр. ЦАГИ, вып. 440 (1939). [6] Kármán and Howarth, Proc. Roy. Soc., A. **164**, 192 (1938).

Translated by V. Levin.

---

* We assume that the relation (26) remains valid for arbitrarily small ρ, but a real sense it has only for ρ large compared to $\frac{\lambda}{L}$, since the assumption (19) itself is natural only for r large in comparison to λ.

DISSIPATION OF ENERGY IN THE LOCALLY ISOTROPIC TURBULENCE

By A. N. KOLMOGOROFF, Member of the Academy

In my note (1) I defined the notion of local isotropy and introduced the quantities

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[u_d(M') - u_d(M)]^2} \\ B_{nn}(r) &= \overline{[u_n(M') - u_n(M)]^2} \end{aligned}\right\} \tag{1}$$

where r denotes the distance between the points M and M', $u_d(M)$ and $u_d(M')$ are the velocity components in the direction $\overline{MM'}$ at the points M and M', and $u_n(M)$ and $u_n(M')$—the velocity components at the points M and M' in some direction, perpendicular to MM'.

In the sequel we shall need the third moments

$$B_{ddd}(r) = \overline{[u_d(M') - u_d(M)]^3}. \tag{2}$$

For the locally isotropic turbulence in incompressible fluid we have the equation

$$4\overline{E} + \left(\frac{dB_{ddd}}{dr} + \frac{4}{r} B_{ddd}\right) = 6\nu \left(\frac{d^2B_{dd}}{dr^2} + \frac{4}{r} \frac{dB_{dd}}{dr}\right) \tag{3}$$

similar to the known equation of Kármán for the isotropic turbulence in the sense of Taylor. Herein $\overline{E}$ denotes the mean dissipation of energy in the unit of time per unit of mass. The equation (3) may be rewritten in the form

$$\left(\frac{d}{dr} + \frac{4}{r}\right)\left(6\nu \frac{dB_{dd}}{dr} - B_{ddd}\right) = 4\overline{E}, \tag{4}$$

and, in virtue of the condition $\frac{d}{dr} B_{dd}(0) = B_{ddd}(0) = 0$, yields

$$6\nu \frac{dB_{dd}}{dr} - B_{ddd} = \frac{4}{5} \overline{E} r. \tag{5}$$

For small r we have, as is known,

$$B_{dd} \sim \frac{1}{15\nu} \overline{E} r^2, \tag{6}$$

i. e.

$$6\nu \frac{dB_{dd}}{dr} \sim \frac{4}{5} \overline{E} r.$$

Thus, the second term on the left-hand side of (5) is for small r infinitesimal in comparison with the first. For large r, on the contrary, the first term may be neglected in comparison with the second, i. e. we may assume that

$$B_{ddd} \sim -\frac{4}{5} \overline{E} r. \tag{7}$$

It is natural to assume that for large r the ratio

$$S = B_{ddd} : B_{dd}^{3/2}, \tag{8}$$

i. e. the *skewness* of the distribution of probabilities for the difference $\Delta u_d = u_d(M') - u_d(M)$, remains constant. Under this assumption we have for large r

$$B_{dd} \sim C\overline{E}^{\frac{2}{3}} r^{\frac{2}{3}} \tag{9}$$

where

$$C = \left(-\frac{4}{5S}\right)^{\frac{2}{3}}. \tag{10}$$

In (1) the relation (9) was deduced from somewhat different considerations *.

In (1) we introduced the *local scale* of turbulence

$$\lambda = \left(\frac{\nu^3}{\overline{E}}\right)^{\frac{1}{4}} \tag{11}$$

and established the assumption that

$$\left.\begin{aligned} B_{dd}(r) &= \sqrt{\nu\overline{E}}\,\beta_{dd}\left(\frac{r}{\lambda}\right) \\ B_{nn}(r) &= \sqrt{\nu\overline{E}}\,\beta_{nn}\left(\frac{r}{\lambda}\right) \end{aligned}\right\} \tag{12}$$

where β_{dd} and β_{nn} are universal functions, for which for small ρ

$$\beta_{dd}(\rho) \sim \frac{1}{15}\rho^2, \quad \beta_{nn}(\rho) \sim \frac{2}{15}\rho^2 \tag{13}$$

and for large ρ

$$\beta_{dd}(\rho) \sim C\rho^{\frac{2}{3}}, \ \beta_{nn}(\rho) \sim \frac{4}{3}C\rho^{\frac{2}{3}}. \tag{14}$$

In the case of isotropic turbulence in the sense of Taylor the laws of locally isotropic turbulence must hold for distances, considerably less than the *integral scale* L of turbulence [as regards its rational definition cf. my note (3)]. The correlation coefficients

$$\left.\begin{aligned} R_{dd}(r) &= (\overline{u_d(M')\,u_d(M)}) : b \\ R_{nn}(r) &= (\overline{u_n(M')\,u_n(M)}) : b \end{aligned}\right\} \tag{15}$$

where b is the mean value of the square of the velocity components, are here connected with $B_{dd}(r)$ and $B_{nn}(r)$ by the relations

$$\left.\begin{aligned} B_{dd} &= 2b(1 - R_{dd}) \\ B_{nn} &= 2b(1 - R_{nn}) \end{aligned}\right\}. \tag{16}$$

In virtue of (16) and (12) we shall have for r, small in comparison with L,

$$\left.\begin{aligned} 1 - R_{dd} &\sim \frac{\sqrt{\nu\overline{E}}}{2b}\beta_{dd}\left(\frac{r}{\lambda}\right) \\ 1 - R_{nn} &\sim \frac{\sqrt{\nu\overline{E}}}{2b}\beta_{nn}\left(\frac{r}{\lambda}\right) \end{aligned}\right\}. \tag{17}$$

If r is small in comparison with L, but large in comparison with λ, we have in virtue of (14) and (11)

$$1 - R_{dd} \sim \frac{1}{2}C\overline{E}^{\frac{2}{3}} b^{-1} r^{\frac{2}{3}}, \tag{18'}$$

$$1 - R_{nn} \sim \frac{2}{3}C\overline{E}^{\frac{2}{3}} b^{-1} r^{\frac{2}{3}}. \tag{18''}$$

---

* A. Obukhoff has found the relation (9) independently by computing the balance of the energy distribution of pulsations over the spectrum [cf. (2)].

The formulae (18) enable us to determine the constant C from the experimental data. The most accurate measurements of the correlation coefficients R_{dd} and R_{nn} were carried out by Dryden, Schubauer, Mock and Scramstad in their paper ([4]). Rewriting (18″) in the form

$$1 - R_{nn} \sim 2C\,(kr)^{\frac{2}{3}}, \quad k = \overline{E} : (3b)^{\frac{3}{2}}, \tag{19}$$

I calculated from the empirical formula (17) of the paper ([4]) (putting in the denotations of ([4]) $b = \sqrt{\overline{u^2}}$, $\overline{E} = \frac{3}{2} U \frac{d\sqrt{\overline{u^2}}}{dx}$) the values of the coefficient k, corresponding to the turbulence at the distance of $40\,M$ from the grid with the width of mesh M equal to 1″, $3^1/_4$″ and 5″:

M in inches	1	$3^1/_4$	5
k in cm^{-1}	0.197	0.065	0.042

With these values for k the graphs in Fig. 5 of ([4]), taking into account the wire-length correction, show for values of r, not too large in comparison with L, a good agreement with formula (19) for

$$C = \frac{3}{2}. \tag{20}$$

The local scale λ is, in the circumstances of experiments described in ([4]), so small that deviations from the relations (18) cannot be observed for small r *.

The curves in Fig. 28 of ([3]) cannot be directly used for the determination of C, since the wire-length correction is not introduced into them. They confirm, however, for r, small in comparison with L, the relation

$$(1 - R_{nn}) : (1 - R_{dd}) = \frac{4}{3} \tag{21}$$

following from (18), with a satisfactory accuracy.

If we take into account the wire-length correction for the ratio (21), in ([4]) is obtained the value 1.28 [cf. ([4]), p. 29], which is also sufficiently close to the theoretical value 4/3, if we consider the limited accuracy of the experiment.

Received
30.IV.1941.

REFERENCES

[1] A. Kolmogoroff, C. R. Acad. Sci. URSS, XXX, 301 (1941). [2] A. Obukhoff, ibid., XXXII, No. 1 (1941). [3] A. Kolmogoroff, ibid., XXXI (1941). [4] Rep. NACA, No. 581 (1937). [5] M. Millionshtchikov, C. R. Acad. Sci. URSS, XXII, 241 (1939).

Translated by V. Levin.

---

* In connexion with this we may remark that the attempt of application to the observations of ([4]) of the theory of isotropic turbulence, neglecting the third moments, undertaken by Millionshtchikov in ([5]), must be considered as based on a misunderstanding. It is easily verified that under the circumstances of observations in ([4]), in the equations connecting the second moments with the third ones [e. g. in equation (3)] the terms with the second moments are *considerably smaller*, than the terms with the third moments.

In Fig. 3 of ([5]) the comparison of the theoretical curve for R_{nn} (obtained under neglection of the third moments) with the experimental data of ([4]) is carried out in the right way, since: 1) the used data refer to a definite spectral component of the pulsations, and not to the total pulsations, and 2) the experimental data do not correspond to the condition $8\ \nu t = 7.56$, by which the theoretical curve is determined. In his later papers Millionshtchikov himself gave, by means of rather fine considerations, using the third and fourth moments, an estimation of the error resulting from the neglection of the third moments.

PROGRESS IN THE STATISTICAL THEORY OF TURBULENCE[1]

By

THEODORE VON KÁRMÁN

California Institute of Technology
Pasadena, California

The fundamental notion of statistical mean values in fluid mechanics was first introduced by Reynolds (1894). In his description of the turbulent motion he started from the molecular disorder. By considering the mean value and correlations in the molecular motion, as it was shown before, one obtains the equation of Stokes-Navier (1845, 1822). The solutions of these equations were designated by Reynolds as "mean motion." The second step in Reynolds' development is the consideration of the turbulent fluctuation. He assumes, for instance, that the apparently parallel flow in a cylindrical tube consists of a fluctuating motion characterized by the fact that the motion, at every instant, satisfies the Stokes-Navier equation. The parallel motion obtained by the formation of average values is called the "mean mean motion."

Reynolds' most important contributions were the definition of the mean values for the so-called Reynolds stresses and the recognition of the analogy between momentum transfer, transfer of heat and matter in the turbulent motion.

In the decades following Reynolds' discoveries, turbulence theory was directed to find semiempirical laws for the mean mean motion by methods borrowed from the kinetic theory of gases, *i. e.*, from the theory of the mean motion.

Prandtl's (1925, 1927) ideas on momentum transfer and Taylor's (1915, 1932) suggestions concerning vorticity transfer belong to the most important contributions of this period. But I believe that my formulation of the problem (1930) by the application of the similarity principle has the merit of being more general and independent of the methods of the kinetic theory of gases. This theory of mine led to the discovery of the logarithmic law of velocity distribution for the case of homologous turbulence. I call the turbulence homologous if the

[1] Presented at the Heat Transfer and Fluid Mechanics Institute, Los Angeles, California, June 23, 1948.

distribution of turbulence fluctuations and their correlations are identical at every point of the field. They differ only in scale.

The next important step was the definition of isotropic turbulence by Taylor (1935, 1936). Apparently the case of homologous turbulence, *i. e.*, the shear motion, is too complicated for a fundamental attack by statistical methods. It was Taylor's fortunate idea to simplify the problem by the consideration of a uniform and isotropic field of turbulent fluctuation. Since such a field can be realized at least approximately in the wind tunnel, the possibility was given for an experimental check of the statistical ideas.

The next period in the development of the theory of turbulence was devoted to the analysis of the quantities which are accessible to measurements in a wind tunnel stream. These quantities are the correlation functions and the spectral function. The general mathematical analysis of the correlations was executed by Howarth and myself (von Kármán and Howarth, 1938). One has to consider five scalar functions $f(r)$, $g(r)$, $h(r)$, $k(r)$, $l(r)$. These functions determine all double and triple correlations between arbitrary velocity components observed at two points because of the tensorial character of the correlations. The two scalar functions for the double correlations are defined as follows:

$$f(r) = \frac{\overline{u_1(x_1, x_2, x_3)\, u_1(x_1 + r, x_2, x_3)}}{\overline{u_1^2}},$$
$$g(r) = \frac{\overline{u_1(x_1, x_2, x_3)\, u_1(x_1, x_2 + r, x_3)}}{\overline{u_1^2}}. \tag{1}$$

Because of the continuity equation for incompressible fluids, $g = f + \frac{r}{2}\frac{df}{dr}$. For the same reason the triple correlations h, k, and l can be expressed by one of them, *i. e.*, by

$$h(r) = \frac{\overline{[u_1(x_1, x_2, x_3)]^2\, u_1(x_1 + r, x_2, x_3)}}{[\overline{u_1^2}]^{3/2}}. \tag{1a}$$

In addition, we also deduced a differential equation from the Stokes-Navier equation which gives the relation between the time derivative of the function f and the triple correlation function h.

$$\frac{\partial}{\partial t}(\overline{u^2}f) + 2[\overline{u^2}]^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{2}$$

We discussed this equation in two special cases:

a) Small Reynolds Number—in this case the triple correlations can be neglected and one obtains a self-preserving form for the double correlation function as function of r/λ, where λ is defined by the relation

$$\frac{d\overline{u^2}}{dt} = -\,10\nu\,\frac{\overline{u^2}}{\lambda^2}\,. \tag{3}$$

b) Large Reynolds Number—in this case the terms containing the viscosity can be neglected for not too small values of r, and the functions f and h are assumed to be functions of the variable r/L, L being a length characterizing the scale of turbulence. The hypothesis of self-preserving correlation function leads to the following special results. One can consider three simple cases:

1. L = constant; then we have $\overline{u^2} \sim t^{-2}$ (Taylor).
2. Loitsianskii (1939) has shown that if the integral $\overline{u^2}\int_0^\infty f(r)\,r^4 dr$ exists, it must be independent of time and consequently $\overline{u^2}L^5$ = constant. Then $\overline{u^2} \sim t^{-10/7}$, $L \sim t^{2/7}$.
3. If the self-preserving character is extended to all values of r, *i. e.*, also near $r = 0$, one has $\overline{u^2} \sim t^{-1}$, $L \sim t^{1/2}$ (Dryden, 1943).

On the other hand, Taylor (1938) introduced a spectral function for the energy passing through a fixed cross section of a turbulent stream as the Fourier transform $\mathfrak{F}_0(n)$ of the correlation function $f(r)$. The relations between $\mathfrak{F}$ and f are given by the following equations:

$$\begin{aligned} f(r) &= \int_0^\infty \mathfrak{F}_0(n)\cos\frac{2\pi n r}{U}\,dn\,, \\ \mathfrak{F}_0(n) &= \frac{4\overline{u_1^2}}{U}\int_0^\infty f(r)\cos\frac{2\pi n r}{U}\,dr\,. \end{aligned} \tag{4}$$

In these equations n is the frequency of the fluctuation of the uniform velocity U as function of time. Relative to the stream, $\mathfrak{F}_0(n)$ can be replaced by $\mathfrak{F}_1(\kappa_1)$, where $\kappa_1 = \frac{2\pi n}{U}$, *i. e.*, the wave number of the fluctuation, measured in the x_1-direction.

It is seen that in this period of the development of the turbulence theory the analytical and experimental means for the study of isotropic turbulence were clearly defined, but with the exception of the case of very small Reynolds numbers no serious attempt was made to find the laws for the shapes of either the correlation or the spectral functions. I believe this is the principal aim of the period in which we find ourselves at present. Promising beginnings were made by Kolmogoroff (1941), Onsager (1945), Weizsäcker (1946), and Heisenberg (1946). I do not want to follow the special arguments of these authors. Rather, I want to define the problem clearly and point out the relations between assumptions and results.

First, we will assume that the three components of the velocity in an homogeneous isotropic turbulent field, at any instant, can be developed in the manner of Fourier's integrals

$$u_i = \iiint_{-\infty}^{\infty} Z_i(\kappa_1, \kappa_2, \kappa_2, t) e^{i(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)} \, d\kappa_1 d\kappa_2 d\kappa_3 \,. \tag{5}$$

Second, the intensity of the turbulent field shall be characterized by the quadratic mean value $\overline{u_i^2}$ (level of the turbulence). Also, there exists a function $\mathfrak{F}(\kappa)$, such that $[\overline{u_i^2}]_0^\kappa = \int_0^\kappa \mathfrak{F}(\kappa')\, d\kappa'$, where the symbol $[\overline{u_i^2}]_0^\kappa$ means a partial mean value of the square of the velocity, the averaging process being restricted for such harmonic components whose wave numbers κ_1, κ_2, κ_3 satisfy the relation

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \leqq \kappa^2 \,.$$

If such a function exists, it is connected with the spectral function of Taylor $\mathfrak{F}_1(\kappa_1)$ by the relation

$$\mathfrak{F}_1(\kappa_1) = \tfrac{1}{4} \int_{\kappa_1}^{\infty} \frac{1}{\kappa^3} \mathfrak{F}(\kappa) \, (\kappa^2 - \kappa_1^2) \, d\kappa \,. \tag{6}$$

This relation was found by Heisenberg (1946). It expresses the geometrical fact that all oblique waves (Fig. 1) with wave length

$$\frac{2\pi}{\kappa} < \frac{2\pi}{\kappa_1}$$

necessarily contribute in the one-dimensional analysis to the waves with wave length

$$\frac{2\pi}{\kappa_1} \,.$$

Third, it is evident that there must be an equation for the time derivative of $\mathfrak{F}(\kappa)$ which corresponds to the equation for the time derivative of $f(r)$ which has been found by Howarth and myself.

The physical meaning of this equation is evident. Let us start from the energy equation

$$\frac{1}{2}\frac{\partial \overline{u_i^2}}{\partial t} + \overline{\left(u_i u_j + \delta_{ij}\frac{p}{\rho}\right)\frac{\partial u_i}{\partial x_j}} = \nu \overline{\frac{\partial^2 u_i}{\partial x_j^2} u_j} . \tag{7}$$

The right side represents the energy dissipation by viscous forces. The second term on the left side is the work of the Reynolds stresses; it represents a transfer of energy without actual dissipation. Our problem is to find $\frac{\partial \mathfrak{F}}{\partial t}$ by Fourier analysis and averaging process. One finds the contribution of the viscous forces to be equal to $-2\nu\,\mathfrak{F}(\kappa)\kappa^2$. Hence we write formally

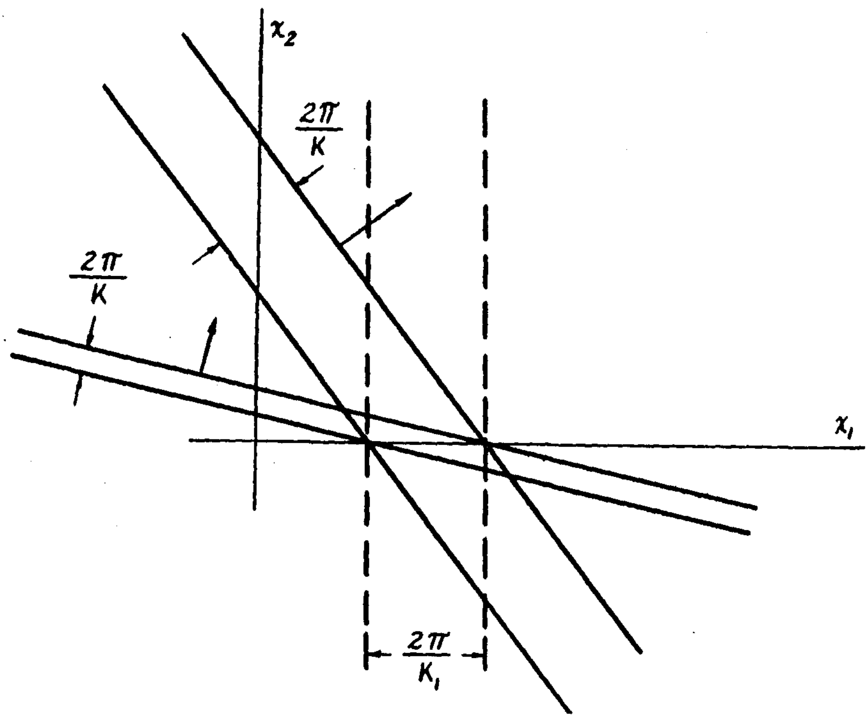


Figure 1. Contribution of oblique waves to plane waves in direction of x_1.

$$\frac{\partial \mathfrak{F}}{\partial t} + \mathfrak{W}_\kappa = -2\nu\kappa^2\,\mathfrak{F}(\kappa) . \tag{8}$$

Here $\mathfrak{W}_\kappa d\kappa$ is the balance for the energy contained in harmonic components contained in the interval $d\kappa$; obviously $\int_0^\infty \mathcal{W}_\kappa d\kappa = 0$. C. C. Lin (personal communication) has shown that $\mathcal{W}_\kappa = 2\mathcal{H}\kappa^2$, where $\mathcal{H}(\kappa) = 2\,(\kappa^2\mathcal{H}_1''(\kappa) - \kappa\mathcal{H}_1'(\kappa))$ and

$$\mathcal{H}_1(\kappa) = \frac{2\overline{u^2}^{3/2}}{\pi} \int_0^\infty h(r) \frac{\sin(\kappa r)}{\kappa}\, dr\,.$$

Unfortunately this relation does not help, as far as the determination of f and h is concerned. For example, if one expresses h in terms of f by means of the Kármán-Howarth equation, calculates $\mathcal{W}_\kappa$ and then substitutes the result in equation (8), one obtains an identity. It appears that at the present time one needs some additional physical assumption.

Fourth, we assume that $\mathcal{W}_\kappa$ can be expressed in the form;

$$\mathcal{W}_\kappa = \int_0^\infty \Theta\{\mathcal{F}(\kappa), \mathcal{F}(\kappa'), \kappa, \kappa'\}\, d\kappa'\,. \tag{9}$$

The physical meaning of this assumption is the existence of a transition function for energy between the intervals $d\kappa$ and $d\kappa'$ which depends only on the energy density and the wave numbers of the two intervals. It follows from this definition that by interchanging κ and κ' one has

$$\Theta\{\mathcal{F}(\kappa), \mathcal{F}(\kappa'), \kappa, \kappa'\} = -\Theta\{\mathcal{F}(\kappa'), \mathcal{F}(\kappa), \kappa', \kappa\}\,. \tag{10}$$

It must be noted that our assumption probably cannot be exact. It is very probable that the values of $\mathcal{F}$ for the difference and the sum of κ and κ' also enter in the transition function. I believe that the assumption gives a fair approximation when κ and κ' are very different, but it is certainly untrue if κ and κ' are nearly equal.

Fifth, we furthermore specify the function Θ in the following way;

$$\Theta = -C\,\mathcal{F}^{\alpha}(\kappa)\,\mathcal{F}^{\alpha'}(\kappa')\,\kappa^{\beta}\kappa'^{\beta'}\,; \quad C = \text{const.} \tag{11}$$

It follows from dimensional reasoning that

$$\alpha + \alpha' = 3/2\,, \qquad \beta + \beta' = 1/2\,.$$

As a result of the sequence of assumptions given above, we obtain the equation;

$$\frac{\partial \mathcal{F}}{\partial t} = C\left[\mathcal{F}^{\alpha}\kappa^{\beta}\int_0^{\kappa} \mathcal{F}^{3/2-\alpha}(\kappa')\kappa_1{}^{1/2-\beta}d\kappa' - \mathcal{F}^{3/2-\alpha}\kappa^{1/2-\beta}\int_{\kappa}^{\infty} \mathcal{F}^{\alpha}(\kappa')\kappa'^{\beta}d\kappa'\right] - 2\nu\kappa^2\mathcal{F}\,. \tag{12}$$

Obviously, if $\mathfrak{F}$ is known for $t = 0$, equation (12) determines the values of $\mathfrak{F}$ for all times. If one neglects the first term on the left side, which represents the decay of turbulence, and chooses the specific values $\alpha = 1/2$, $\beta = -3/2$, one arrives at the theory proposed by Heisenberg (1946).

Let us consider the case of large Reynolds numbers but assume that κ is not so large that the term containing the viscosity coefficient becomes significant. Let us assume also that the first term on the left side is small by comparison to the second term. Physically, this means that the energy entering in the interval $d\kappa$ is equal to the energy which leaves the interval. Then one has the relation:

$$\mathfrak{F}^{\alpha}\kappa^{\beta} \int_0^{\kappa} \mathfrak{F}^{3/4-\alpha}(\kappa')\kappa'^{1/2-\beta}d\kappa' = \mathfrak{F}^{3/2-\alpha}\kappa^{1/2-\beta} \int_{\kappa}^{\infty} \mathfrak{F}^{\alpha}(\kappa')\kappa'^{\beta}d\kappa' . \tag{13}$$

This equation is satisfied by the solution $\mathfrak{F}(\kappa) \sim \kappa^{-5/3}$, as one can easily see by substitution in equation (13). This result is independent, evidently, of the special choice of α and β. That is the reason why it was independently found by Onsager (1945), Kolmogoroff (1941) and Weizsäcker (1946). It is essentially a consequence of dimensional considerations. Let us now stay with the case of large Reynolds numbers by neglecting again the viscosity term while retaining the first term on the left side. In other words, we consider the actual process of decay at large Reynolds numbers. Let us assume that $\mathfrak{F}$ is a function of a nondimensional variable κ/κ_0, when κ_0 is a function of time. This assumption is equivalent to our former assumption that $f(r)$ is a function of r/L; *i. e.*, we assume that $\mathfrak{F}$ and f preserve their shapes during the decay. Evidently $\kappa_0 \sim 1/L$. Then the function can be written in the form

$$\mathfrak{F}(\kappa) = \frac{\overline{u^2}}{\kappa_0} \Phi\left(\frac{\kappa}{\kappa_0}\right).$$

Then with $\kappa/\kappa_0 = \xi$ and

$$\frac{\partial \mathfrak{F}}{\partial t} = \frac{\Phi}{\kappa_0}\frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\Phi\frac{d\kappa_0}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\Phi'(\xi)\,\xi\frac{d\kappa_0}{dt},$$

equation (12) becomes

$$\left(\frac{1}{\kappa_0}\frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2}\frac{d\kappa_0}{dt}\right)\Phi - \frac{\overline{u^2}}{\kappa_0^2}\frac{d\kappa_0}{dt}\Phi'\xi + \mathfrak{W}_{\kappa} = 0, \tag{14}$$

where

$$\mathfrak{W}_{\kappa} = -C\,[\overline{u^2}]^{3/2}\,[\Phi^{\alpha}\xi^{\beta}\int_0^{\xi}\Phi(\xi')^{3/2-\alpha}\xi'^{1/2-\beta}d\xi' - \Phi^{3/2-\alpha}\xi^{1/2-\beta}\int_{\xi}^{\infty}\Phi(\xi')^{\alpha}\,\xi'^{\beta}d\xi'] .$$

According to Loitsianskii's (1939) results,

$$\frac{1}{\overline{u^2}} \frac{d\overline{u^2}}{dt} = \frac{5}{\kappa_0} \frac{d\kappa_0}{dt},$$

and one obtains the equation

$$\xi^5 (\xi^{-4}\Phi)' = -5C \frac{[\overline{u^2}]^{3/2}}{\frac{d\overline{u^2}}{dt}} \left[\Phi^\alpha \xi^\beta I_0{}^\xi - \Phi^{3/2-\alpha} \xi^{1/2-\beta} I_\xi{}^\infty \right], \qquad (15)$$

where

$$I_0{}^\xi = \int_0^\xi \Phi(\xi')^{3/2-\alpha} \xi'^{1/2-\beta} d\xi' \; ; \quad I_\xi{}^\infty = \int_\xi^\infty \Phi(\xi')^\alpha \xi'^\beta d\xi' .$$

Let us assume that $4\alpha + \beta < 5/2$ as, for example, in the case of Heisenberg (1946). Then for small values of ξ the right side of equation (15) is small in comparison with the term on the left side, and one has

$$\Phi(\xi) \cong \text{const}\ \xi^4 .$$

If $4\alpha + \beta > 5/2$, $\mathfrak{F}$ begins with a lower power of κ than κ^4, one can show that the integral $\int_0^\infty r^4 f(r) dr$ does not converge, so that Loitsianskii's result is incorrect. I should like to investigate this second case in a later work. Let us assume, for the time being, that Loitsianskii's result is correct; therefore the first case prevails. Then it follows that $\mathfrak{F}$ or Φ behaves as $(\kappa/\kappa_0)^4$ for small values of κ and is proportional to $(\kappa/\kappa_0)^{-5/3}$ for large values of κ. For any definite choice of α and β the differential equation (15) can be solved numerically. The result that $\mathfrak{F} \cong \kappa^4$ for small values of κ was also found in a different way by C. C. Lin (personal communication).

For the time being I propose an interpolation formula as follows:

$$\Phi(\xi) = \text{const.} \frac{\xi^4}{(1 + \xi^2)^{17/6}} . \qquad (16)$$

This interpolation formula represents correctly $\Phi(\xi)$ for small and large values of ξ and has the advantage that all calculations can be carried out analytically by use of known functions. The results are as follows:

$$\mathfrak{F}(\kappa/\kappa_0) = \text{const.}\frac{(\kappa/\kappa_0^4)}{[1 + (\kappa/\kappa_0)^2]^{17/6}},$$

$$\mathfrak{F}_1(\kappa_1/\kappa_0) = \text{const.}\frac{1}{[1 + (\kappa_1/\kappa_0)^2]^{5/6}},$$

$$f(\kappa_0 r) = \frac{2^{2/3}}{\Gamma(1/3)}(\kappa_0 r)^{1/3} K_{1/3}(\kappa_0 r),$$

$$g(\kappa_0 r) = \frac{2^{2/3}}{\Gamma(1/3)}(\kappa_0 r)^{1/3}\left[K_{1/3}(\kappa_0 r) - \frac{\kappa_0 r}{2} K_{-2/3}(\kappa_0 r)\right]. \qquad (17)$$

The K's are Bessel functions with imaginary argument. For small values of $\kappa_0 r$,

$$f(\kappa_0 r) = 1 - \frac{\Gamma(2/3)}{\Gamma(4/3)}\left(\frac{\kappa_0 r}{2}\right)^{2/3}, \qquad (18)$$

as suggested by Kolmogoroff's theory.

I have compared these results with the measurements of Liepmann, Laufer and Liepmann (1948) carried out at the California Institute of Technology with the financial assistance of the National Advisory Committee for Aeronautics.[2] These observations were made in the 10-foot wind tunnel of the Guggenheim Aeronautical Laboratory, using a grid whose mesh size was $M = 4''$. The measurements were made at a distance $x = 40.4M$ from the grid. Fig. 2 shows the comparison of calculated and measured values for the spectral function $\mathfrak{F}_1(\kappa_1)$. It has to be taken into account that the observed values of $\mathfrak{F}_1(\kappa_1)$ have large scatter; the deviation for high values of κ_1 corresponds to the beginning influence of viscosity. Fig. 3 gives a comparison between measured and calculated values of the correlation function $g(r)$. This function is chosen because the observations are more accurate than in any other case. It is seen that the agreement is almost too good in view of the assumptions made above. One must remark that there is only one arbitrary constant in the formula for g, *viz.*, the constant κ_0 which determines the scale of the turbulence. It is true that some of the data of Liepmann, *et al.* (1948) do not show such a good agreement. The agreement is excellent for values of g larger than 0.1, but after that the measured values are higher than the calculated ones. Possibly some oscillations existing in the wind tunnel stream were interpreted as turbulence, or the turbulence is not quite isotropic.

[2] The N. A. C. A. has kindly granted permission for these data to be used here before appearing in its official publications.

I believe that the merits of my deduction are:

a) The assumptions involved are exactly formulated;
b) The specific assumptions of Heisenberg's theory concerning the transition function are not used;
c) The actual process of decay is considered;
d) The analysis is extended to the lower end of the turbulence spectrum.

Concerning the case of large values of κ (small values of r), Kovásznay (1948) introduced an interesting assumption which is more

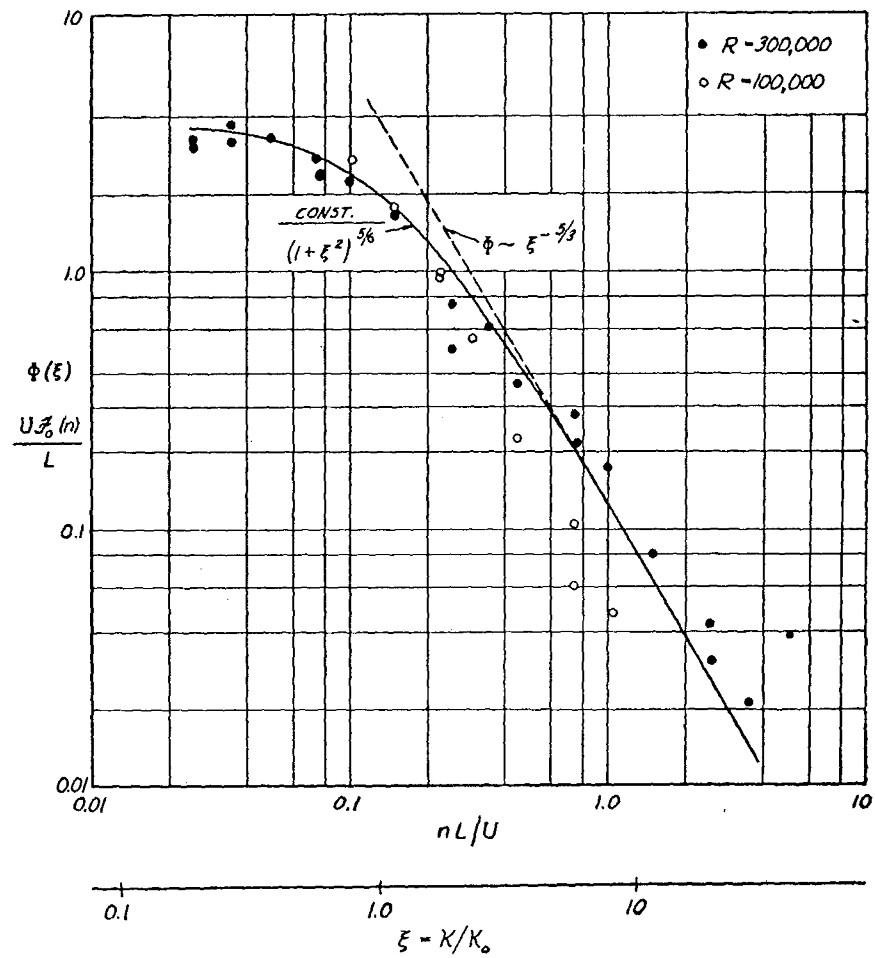


Figure 2. Comparison of observed and computed values of the frequency spectrum.

restricting than my fourth assumption. Obviously $\int_0^\kappa \mathcal{W}_\kappa d\kappa$ is the total energy transferred by the Reynolds stresses from the interval $0 \to \kappa$ to the interval $\kappa \to \infty$. Kovásznay assumes—following Kolmogoroff's (1941) arguments—that this quantity is a function of $\mathfrak{F}(\kappa)$ and κ only. Then for dimensional reasons

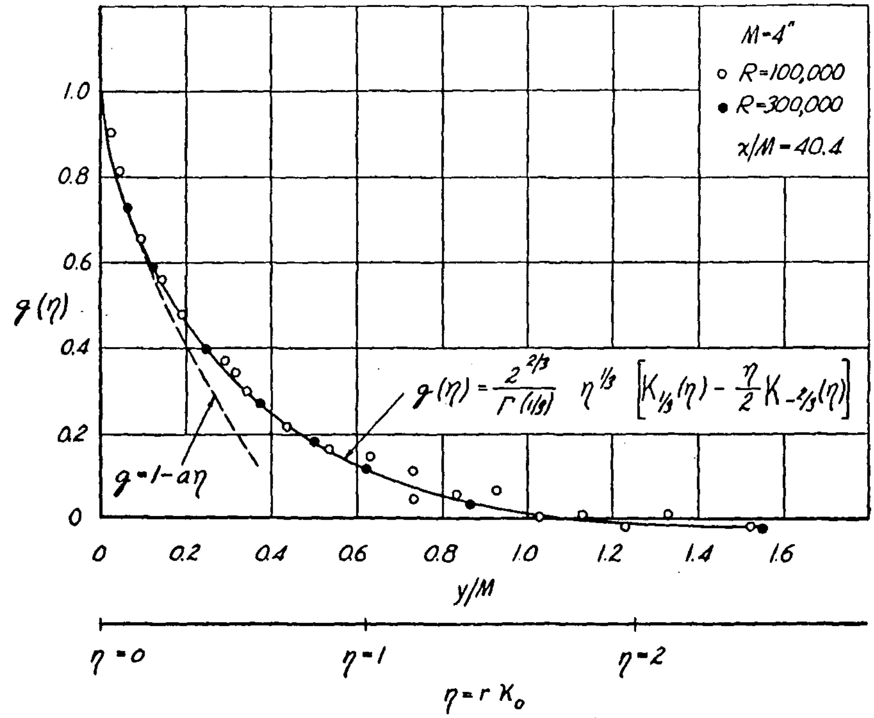


Figure 3. Comparison of observed and computed values of the correlation function $g(r)$. The Reynolds number is based on the stream velocity and the mesh size.

$$\int_0^\kappa \mathcal{W}_\kappa d\kappa = \text{const}\ \mathfrak{F}^{3/2}\kappa^{5/2}.$$

This assumption appears to be correct for large values of κ. However, when the assumption is extended to the range of small values of κ, and one substitutes $\mathcal{W}_\kappa$ in equation (8), one can calculate easily $\mathfrak{F}(\kappa)$. Neglecting the viscous term, one obtains the relation

$$\mathfrak{F}(\kappa/\kappa_0) = \text{const.} \frac{(\kappa/\kappa_0)^4}{[1 + \mathfrak{F}^{1/2}\,(\kappa/\kappa_0)^{3/2}]^{17/2}}. \tag{19}$$

The right side of equation (19) behaves as my corresponding equation (17) for small and large values of κ/κ_0. It will be interesting to see how far different transition from small to large values influences the accordance with observation.

REFERENCES

DRYDEN, H. L.
1943. A review of the statistical theory of turbulence. Quart. appl. Math., *1:* 7-42.

HEISENBERG, W.
1946. Zur Statistischen Theorie der Turbulenz. Unpublished.

KÁRMÁN, THEODORE VON
1930. Mechanische Ähnlichkeit und Turbulenz. Nachr. Ges. Wiss. Gottingen, Mat.-phys. Kl., pp. 58–76.

KÁRMÁN, THEODORE VON, AND LESLIE HOWARTH
1938. On the statistical theory of isotropic turbulence. Proc. roy. Soc. Lond., (A) *164:* 192–215.

KOVÁSZNAY, L. S. G.
1948. The spectrum of locally isotropic turbulence. Phys. Rev., *73* (9): 1115.

KOLMOGOROFF, A.
1941. The local structure of turbulence in incompressible viscous fluids for very large Reynolds numbers. C. R. (Doklady) Acad. Sci. U. R. S. S., N. S. *30* (4): 301–305.

LIEPMANN, H. W., J. LAUFER AND K. LIEPMANN
1948. On some turbulence measurements behind grids. Final Rep. nat. adv. Comm. Aero., Wash. Contract NAw 5442.

LOITSIANSKII, L. G.
1939. Some basic laws of isotropic turbulent flow. Rep. Cent. Aero-Hydrodynam. Inst., Moscow, No. 440. (Transl., Techn. Memo. nat. adv. Comm. Aero., Wash., 1079: 1–36, 1945.)

NAVIER, M. H.
1822. Sur les lois du mouvement des fluides. Mem. Acad. Sci., Paris, *6:* 389–440.

ONSAGER, LARS
1945. The distribution of energy in turbulence. Phys. Rev., (II) *68:* 286.

PRANDTL, L.
1925. Bericht über Untersuchungen zur ausgebildeten Turbulenz. Z. angew. Math. Mech., *5:* 136–139.
1927. Ueber die ausgebildete Turbulenz. Verh. 2. int. Kongr. techn. Mechanik, Zurich, 1926, pp. 62–74.

REYNOLDS, OSBORNE
1894. On the dynamical theory of incompressible viscous fluids and the determination of the criterion. Proc. roy. Soc. Lond., *56:* 40–45 (Abstr.).

STOKES, G. G.
1845. On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. Trans. Camb. phil. Soc., *8:* 287–319.

TAYLOR, G. I.
1915. Eddy motion in the atmosphere. Philos. Trans., (A) 215: 1–26.
1932. The transport of vorticity and heat through fluids in turbulent motion. Proc. roy. Soc. Lond., (A) *135:* 685–702.
1935. Statistical theory of turbulence. I. Proc. roy. Soc. Lond., (A) *151:* 421–444.
1935. Statistical theory of turbulence. II. Proc. roy. Soc. Lond., (A) *151:* 444–454.
1935. Statistical theory of turbulence. III. Distribution of dissipation of energy in a pipe over its cross-section. Proc. roy. Soc. Lond., (A) *151:* 455–464.
1935. Statistical theory of turbulence. IV. Diffusion in a turbulent air stream. Proc. roy. Soc. Lond., (A) *151:* 465–478.
1936. Statistical theory of turbulence. V—Effect of turbulence on boundary layer. Theoretical discussion of relationship between scale of turbulence and critical resistence of spheres. Proc. roy. Soc. Lond., (A) *156:* 307–317.
1938. The spectrum of turbulence. Proc. roy. Soc. Lond., (A) *164:* 476–490.

WEIZSÄCKER, C. F. VON
1946. Das Spektrum der Turbulenz bei grossen Reynolds' schen Zahlen. Unpublished.

NOTE ON THE LAW OF DECAY OF ISOTROPIC TURBULENCE

By C. C. Lin

Massachusetts Institute of Technology

Communicated by Theodore von Kármán, August 2, 1948

The law of decay of turbulence has been discussed by several authors by making various assumptions. Recently, Batchelor[1] made a summarizing discussion of some of these assumptions and their consequences. The present author has made a different assumption, which seems to be suggested by Kolmogoroff's concept of turbulence at high Reynolds numbers, and yet has not been covered by Batchelor's discussion. The result seems also to be in reasonably good agreement with all experimental data available. This result was originally obtained by a discussion of the spectrum, but a derivation using the correlation function itself is presented here.

According to Kolmogoroff, the double and triple correlation functions $f(r)$ and $h(r)$ for a distance r apart should be of the forms

$$u'^2\{1 - f(r)\} = v^2\beta_2\left(\frac{r}{\eta}\right), \tag{1}$$

$$u'^3h(r) = v^3\beta_3\left(\frac{r}{\eta}\right), \tag{2}$$

where β_2 and β_3 are definite functions of their argument $\xi = r/\eta$ for small values of ξ, at high Reynolds numbers, u' is the intensity of turbulence, and η and v are characteristic measures of length and velocity. Furthermore, η and v are given in terms of the rate of energy dissipation by the following relations

$$\eta = (\nu^3/\epsilon)^{1/4}, \qquad v = (\nu\epsilon)^{1/4}, \qquad \epsilon = -\frac{3}{2}\frac{du'^2}{dt}, \tag{3}$$

where t is the time, and ν is the kinematic viscosity coefficient. At the various stages of the decay process, ϵ changes. Comparing the fields at various stages of decay, one sees that $\beta_2(\xi)$ and $\beta_3(\xi)$ should be definite functions of ξ for small ξ in the course of time, i.e., there is self-preservation of $\beta_2(\xi)$ and $\beta_3(\xi)$.

We shall now show that if one makes the assumption of self-preservation of β_2 and β_3, (3) follows as a consequence. Also, a definite law of decay is obtained which is in agreement with all experimental data so far available.

Substituting (1) and (2) into the equation of von Kármán and Howarth,

$$\frac{\partial}{\partial t}(u'^2f) + 2u'^3\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 2\nu u'^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right),$$

one has

$$\frac{du'^2}{dt} - \frac{dv^2}{dt}\,\beta_2(\xi) + v^2\beta_2'(\xi)\xi\,\frac{1}{\eta}\,\frac{d\eta}{dt} + \frac{2v^3}{\eta}\left\{\beta_3'(\xi) + \frac{4}{\xi}\,\beta_3(\xi)\right\} = \frac{2\nu v^2}{\eta^2}\left\{\beta_2''(\xi) + \frac{4}{\xi}\,\beta_2'(\xi)\right\}.$$

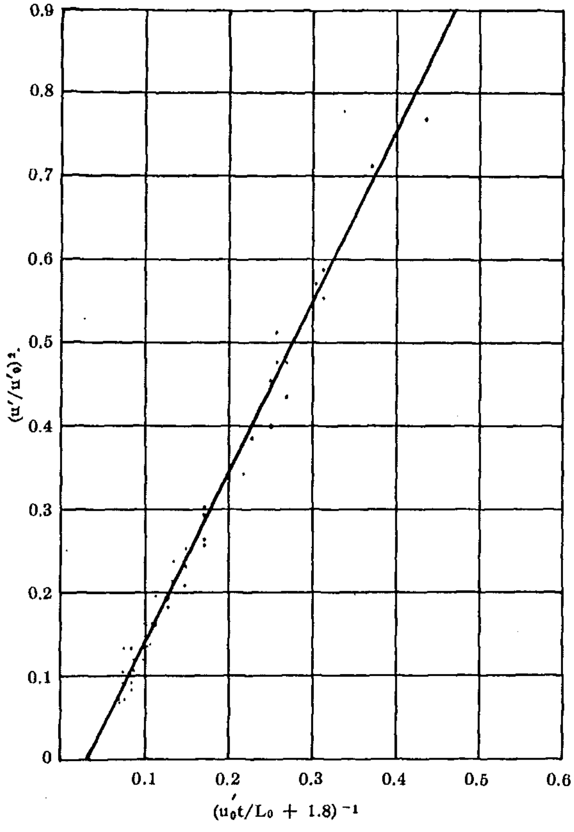


FIGURE 1

Comparison of the present law of decay with experiments.

Thus, the ratios

$$\epsilon \;:\; \frac{dv^2}{dt} \;:\; v^2\,\frac{1}{\eta}\,\frac{d\eta}{dt} \;:\; \frac{v^3}{\eta} \;:\; \nu\,\frac{v^2}{\eta^2}$$

must be constants. From a consideration of the first, the fourth and the fifth terms, (3) follows at once. Including the other two terms, one obtains the additional relations:

does however include the half-power law which is at least approximately satisfied by many of the experiments. On the other hand, Dryden's analysis[2] indicates that there are certain data which do depart from it. One can however take such results into account by the present law. The comparison is shown in figure 1, where one of Dryden's diagrams (von Kármán) is replotted according to the present scheme. This is the case departing most from the half-power law. It is seen that the agreement is good.

A more critical check is to compare the theory with directly measured values of λ, as λ depends on the derivative of u'. Such a comparison is made in figure 2. In making this comparison, the curve is supposed to begin with a slope of $10\nu M/U$ (M being mesh size of the grid, $U = x/t$) and approaches asymptotically to a straight line of slope $4\nu M/U$. These limiting straight lines are shown dotted, where they depart considerably from the experimental values. In fact, both the experimental points and these lines are reproduced from a figure kindly supplied to the author by Mr. Batchelor, based on experimental results of Dr. Townsend. The theoretical curve is then made to join up smoothly with the asymptotic straight line. The agreement is good. It may be noted that there is no freedom in this joining.

The decay process may thus be described as follows. At the beginning of the decay process, when the Reynolds number of turbulence is high, there is self-preservation in the sense of (1) and (2) for small r. As the Reynolds number decreases to very low values, the self-preservation property includes the whole $f(r)$ curve. This is the final stage of the decay process which has been considered by von Kármán and Howarth and investigated very much in detail recently by Batchelor and Townsend. The physical process is much clearer when presented in terms of spectrum. This discussion will be presented elsewhere.

The appearance of the arbitrary additive constant might seem to be a drawback in the theory. Actually, it is a necessity, if one follows the idea of Kolmogoroff. According to this theory, at high Reynolds numbers, the essential decay mechanism is governed by the high-frequency components only. On the other hand, the low-frequency components do contribute to the intensity. For example, if there is a superposed disturbance of low frequencies, the essential decay mechanism should not change. The detailed discussion of this point will be presented later in connection with a study of the energy spectrum. It might be noted that when $\beta = 0$, we have the half-power law of decay, and consequently a self-preservation of $f(r)$ itself. In any case, this is approximately satisfied for the initial stages of the decay process, when $t - t_0$ is small.

[1] Batchelor, G. K., *QuAM*, 6, 97–116 (1948).
[2] Dryden, H. L., *Ibid.*, 1, 28–30 (1943).

$$v^2 = A(t - t_0)^{-1}, \qquad \eta^2 = B\nu(t - t_0), \tag{4}$$

where A, B and t_0 are constants. From this, it can be easily seen that

$$\epsilon = C(t - t_0)^{-2}, \tag{5}$$

and hence

$$u'^2 = \alpha(t - t_0)^{-1} + \beta, \tag{6}$$

where C, α and β are constants. This leads also to the relation

$$\lambda^2 = 10\nu(t - t_0)\{1 + (\beta/\alpha)(t - t_0)\}, \tag{7}$$

for the change of Taylor's microscale λ.

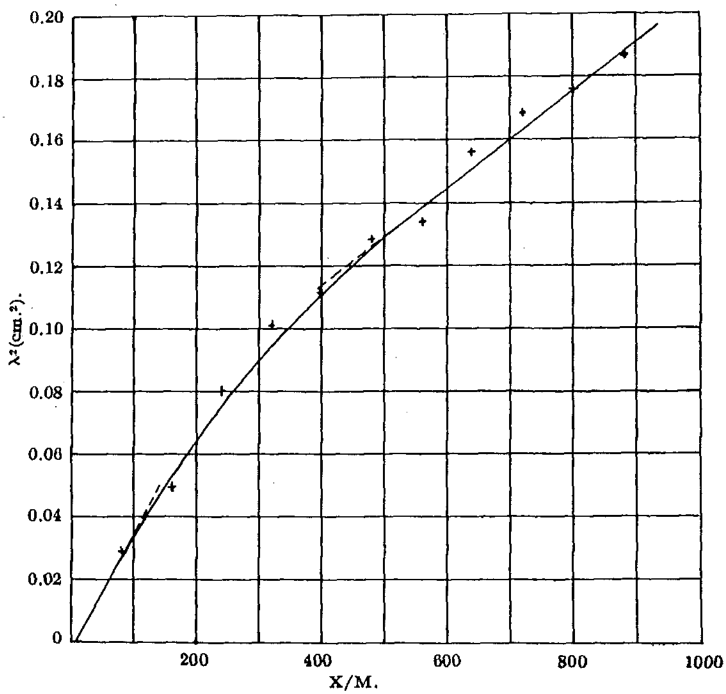


FIGURE 2

Comparison with experiments of the theoretical law of change of Taylor's microscale.

Notice that η^2 increases linearly in t. As η is the smallest scale, this diffusive nature is expected. Equations (5), (6) and (7) are merely consequences of this relation. Taylor's microscale λ does not in general possess the diffusive property any more (except when β is zero). But it can be easily verified that $R_\lambda = u'\lambda/\nu$ and $u'^{-2}R_\lambda$ changes linearly in time.

The law of decay just obtained is different from all previous ones. It

On the Concept of Similarity in the Theory of Isotropic Turbulence

THEODORE VON KÁRMÁN

AND

C. C. LIN

MUCH recent work has been done in the study of isotropic turbulence, particularly from the point of view of its spectrum. But the underlying concept is still the assumption of the similarity of the spectrum during the process of decay, which is equivalent to the idea of self-preservation of the correlation functions introduced by the senior author. It is however generally recognized that the correlation function does change its shape during the process of decay, and hence the concept of self-preservation or similarity must be interpreted with suitable restrictions. Under the limitation to low Reynolds numbers of turbulence, the original idea of Kármán-Howarth has been confirmed. Then the decay consists essentially of viscous dissipation of energy separately in each individual frequency interval. However, when turbulent diffusion of energy, i.e., transfer of energy between frequency intervals, occurs at a significant rate, the interpretation of the decay process and the spectral distribution is quite varied. This can be seen by a comparison of the recent publications of Heisenberg,[1] Batchelor,[2] Frenkiel,[3] and the present authors.[4,5] The purpose of the present paper is an attempt to clarify this situation.

Since some of these discussions are presented in terms of the correlation functions and others in terms of the spectrum, we shall begin by giving a systematic demonstration of the relation between these two theories. This can be easily done by a three-dimensional Fourier transform of the equation for the change of the double correlation tensor:

$$\partial/\partial t(u'^2 R_{ik}) - u'^3(\partial/\partial \xi_j)(T_{ijk} + T_{kji}) = 2\nu u'^2 R_{ik}, \tag{1}$$

[1]W. Heisenberg, Proc. Roy. Soc. London, A **195**, 402 (1948).

[2]G. K. Batchelor, "Recent developments on turbulence research," general lecture at the Seventh International Congress for Applied Mechanics, September, 1948.

[3]F. N. Frenkiel, "Comparison between theoretical and experimental laws of decay of turbulence," presented at the Seventh International Congress for Applied Mechanics, September, 1948.

[4]Th. v. Kármán, Comptes Rendus **226**, 2108 (1948); Proc. Nat. Acad. Sci. **34**, 531 (1948); Sverdrup anniversary volume (1949).

[5]C. C. Lin, Proc. Nat. Acad. Sci. **34**, 540 (1948); "On the law of decay and the spectrum of isotropic turbulence," presented at the Seventh International Congress for Applied Mechanics, September, 1948.

where u'^2 is the mean square of the turbulent of velocity, t is the time, ν is the kinematic viscosity coefficient, and $R_{ik}(\xi_l, t)$ and $T_{ijk}(\xi_l, t)$ are the double and the triple correlation tensor defined by Kármán and Howarth for two points P and P' separated by a space vector ξ_l. By contracting the resultant equation and multiplying it with $4\pi\kappa^2/3$, where κ is the wave number, we obtain the following equation for the change of spectrum:

$$\partial F/\partial t + W = -2\nu\kappa^2 F, \tag{2}$$

where

$$\begin{aligned} F &= (4\pi\kappa^2/3)F_{nn}, \\ F_{ik}(\kappa_l) &= u'^2/(2\pi)^3 \int\int\int R_{ik}(\xi_l, t)e^{i(\kappa_n \xi_n)}d\tau(\xi), \\ W &= (4\pi\kappa^2/3)\cdot 2i\kappa_j W_{njn}, \\ W_{ijk}(\kappa_l) &= u'^3/(2\pi)^3 \int\int\int T_{ijk}(\xi_l, t)e^{i(\kappa_n \xi_n)}d\tau(\xi). \end{aligned} \tag{3}$$

Evaluating these integrals in terms of spherical coordinates in the ξ-space we obtain

$$\begin{aligned} F &= \tfrac{1}{3}\{\kappa^2 F_1''(\kappa) - \kappa F_1'(\kappa)\}, \\ F_1(\kappa) &= 2u'^2/\pi \int_0^\infty f(r, t) \cos\kappa r dr; \\ W &= \tfrac{2}{3}\{\kappa^2 H_1''(\kappa) - \kappa H_1'(\kappa)\}, \\ \kappa H_1(\kappa) &= 2u'^3/\pi \int_0^\infty h(r, t) \sin\kappa r dr. \end{aligned} \tag{4}$$

where $f(r, t)$ and $h(r, t)$ are the double and triple correlation functions satisfying the Kármán-Howarth equation:

$$\partial/\partial t(u'^2 f) + 2u'^3(\partial h/\partial r + 4h/r) = 2\nu u'^2(\partial^2 f/\partial r^2 + (4/r)(\partial f/\partial r)). \tag{5}$$

The relations (4) which connect (2) and (5) have been obtained previously (1947) by the junior author.[6] The function F is essentially identical with Heisenberg's spectral function, whereas F_1 is the spectrum function introduced by Taylor about a decade ago on the basis of a one-dimensional Fourier analysis of wind-tunnel turbulence. The tensor F_{ik} in (3) was introduced and studied by Batchelor and Kampé de Férièt in 1948.

---

[6] C. C. Lin, "Remarks on the spectrum of turbulence," presented at the First Symposium of Applied Mathematics, American Mathematical Society, August, 1947; to appear in the Proceedings of the symposium.

In this note, we shall restrict ourselves to the spectral theory. We propose to analyse the spectrum and its change during the process of decay. We distinguish two extreme cases: (a) the Reynolds number of turbulence is initially very large, and (b) very small initial Reynolds numbers. The latter case is very much simpler, and can be explained in a few words after the first case is investigated. We shall therefore now consider the case of very large Reynolds numbers.

There is no general principle known which would determine the most probable energy distribution over the spectrum. The problem we deal with is not a question of statistical equilibrium in the proper sense. We shall base our investigations on the concept that during the process of decay the spectrum shows a tendency to become similar. Similarity in this case means that the spectrum can be expressed in the form

$$F = U^2 l \varphi(\kappa l), \tag{6}$$

where U is a typical velocity and l a typical length. The problem is to connect these typical quantities with measurable ones, such as the kinematic viscosity ν, Loitsiansky's invariant J_0, and the rate of energy dissipation ϵ. Full similarity would mean that it is possible to express U and l by unique relations for all values of κ and for the whole process of decay. Dealing with the experimental evidences and by dimensional considerations, one readily recognizes that this is not possible. Hence, one looks for a solution which best satisfies the similarity requirement.

All the authors agree in the following picture: the low frequency ranges contain the bulk of energy, while the viscous dissipation is negligible. They furnish energy by the action of inertial forces to the high frequency ranges, where it is converted into heat. Physically, this was seen by Taylor in the early stages of the development of the theory, but Kolmogoroff made these ideas more precise.

Kolmogoroff recognized that based on this physical concept the parameters which determine U and l for the high frequency range are the coefficient of kinomatic viscosity ν and the rate of energy dissipation ϵ. The rate of dissipation in any case is equal to $10\nu u'^2/\lambda^2$, where λ is the microscale in the dissipation mechanism. It is essential in Kolmogoroff's concept that u' and λ do *not* appear explicitly in the similarity analysis. Thus,

$$U = (\nu\epsilon)^{\frac{1}{4}}, \quad l = (\nu^3/\epsilon)^{\frac{1}{4}}. \tag{7}$$

It is reasonable to assume that at the lower end of this κ-range, the spectrum becomes independent of ν. Then necessarily

$$F \sim \epsilon^{\frac{2}{3}} \kappa^{-5/3}, \tag{8}$$

as it was concluded by several authors.

On the other hand, concerning the low frequencies, the junior author has first shown[6] that the lowest frequencies involve a fixed parameter, i.e., Loitsansky's invariant J_0, and indeed that the spectrum is of the form $J_0\kappa$.[4]

This situation clearly shows the infeasibility of one similarity range extending from $\kappa = 0$ to $\kappa = \infty$, and furthermore makes it necessary to consider at least three ranges: that of the lowest frequencies essentially determined by Loitsiansky's invariant, an intermediary range which not necessarily complies with the similarity characteristics of the high range but is significantly influenced by the rate of dissipation ϵ (though not directly by ν), and the high frequency range in the sense of Kolmogoroff.

We therefore introduce three sets of characteristic quantities, namely, V^*, L^*; V, L; and v, η for the lowest, the medium, and the high frequency ranges respectively. The problem is how to connect them with each other and with other physical quantities.

Equation (7) gives

$$v = (\nu\epsilon)^{1/4}, \quad \eta = (\nu^3/\epsilon)^{1/4}. \tag{7a}$$

For the lowest and the medium ranges, we postulate the existence of a parameter (in general variable with time) which is common to these ranges, and governs the complicated mechanism of energy transfer in these low frequency ranges as the viscosity governs transfer of energy into heat in the high range. The formal statement of this hypothesis is

$$V^*L^* = VL = D. \tag{9}$$

We may call this parameter D the transfer coefficient or eddy diffusion coefficient of the turbulence mechanism. Obviously, it has a significance only when turbulent diffusion is active.

By introducing this hypothesis, we come to the following picture. In the lowest range, as it was pointed out, the invariant J_0 has a decisive influence. Hence, the characteristic parameters must be determined by J_0 and D. In the medium range, they must depend not only on D, but also on ϵ, since this range supplies the energy to be dissipated in the high frequency range. Thus, we have

$$V = (D\epsilon)^{1/4}, \quad \text{and} \quad L = (D^3/\epsilon)^{1/4}, \tag{10}$$

for the medium range, analogous to (7a), and

$$V^* = (D^5/J_0)^{1/3} \quad \text{and} \quad L^* = (J_0/D^2)^{1/3}, \tag{11}$$

for the range of lowest frequencies.

The three ranges, with characteristic quantities (7a), (10), and (11) appear clearly separated when their scales are much different from each other.

Thus, when

$$\eta/L = (D/\nu)^{-3/4} \ll 1, \tag{12}$$

the high and intermediate ranges appear clearly defined over significant parts of the whole spectrum. In between, there is a transition range depending only on the parameter ϵ common to both ranges. From dimensional arguments, we have for this transition range

$$F \sim \epsilon^{2/3} \kappa^{-5/3} \tag{13}$$

in accordance with Kolmogoroff's result for high Reynolds numbers. We shall see later that the condition (12) warrants the high value for the Reynolds number. Similarly, when

$$L/L^* = (T/T^*)^{1/2} \ll 1, \quad (T = L/V, \quad T^* = L^*/V^*), \tag{14}$$

we have a transition range between the low and medium frequencies. In this transition range, the spectrum depends only on D. Again, dimensional arguments show that there

$$F \sim D^2 \kappa. \tag{15}$$

The physical significance of (14) will be explained below.

The exact behavior in the medium range will be determined by the fact which of the fixed parameters ν or J_0 has the predominating influence. We believe that we can arrive at a satisfactory description of the actual process by assuming that a change over takes place. We may divide the process into three stages: (I) the early stage, in which we shall see that $\eta : L$ = constant, (II) the intermediate stage, in which we shall see that $L : L^*$ constant, and (III) the final stage, in which the distinction of several scales is impossible. This last case is the well-understood case of complete similarity at extremely low Reynolds numbers.

(I) *The early stage.* The turbulence field is actually created by some mechanism, natural or artificial, which produce individual eddies. Apparently, these eddies converge toward a kind of statistical balance through exchange of energy. The first period after homogeneity and isotropy are established shall be designated the early stage of the decay process. Experimental evidence on the decay law in these early stages shows that the similarity prevailing at high frequencies extends to the medium range. This statement is identical with the conclusion reached by the junior author,[5] assuming that a perfect similarity of the correlation function exists with the exception of correlations of points between very large distances. Accordingly V and L are constant multiples of v and η. Hence,

$$D = V^* L^* = VL \sim v\eta \tag{16}$$

and because $v\eta = \nu$, D is a constant. On the other hand,

$$V^{*2}L^{*5} = J_0. \tag{17}$$

Thus, in the early stage, the spectral function for the lowest frequency range appears to be independent of time. This fixed range extends as far as the linear part of the spectrum described by (15).

The spectrum is thus as shown in Fig. 1 (after Batchelor). By integration, one arrives at

$$u'^2 = \int_0^\infty F(\kappa)d\kappa = \text{const.}\, V^2 - u_D^2, \tag{18}$$

where u_D^2 is represented by the shaded area. By computing $\epsilon = -du'^2/dt$ and using the relation (10), we see that $V^2 \sim t^{-1}$, and

$$u'^2 = \frac{D_0}{10}t^{-1} - u_D^2, \quad \lambda^2 = 10\nu t\{1 - 10u_D^2 t/D_0\}, \quad R_\lambda = R_{\lambda 0}\{1 - 10u_D^2 t/D_0\}, \tag{19}$$

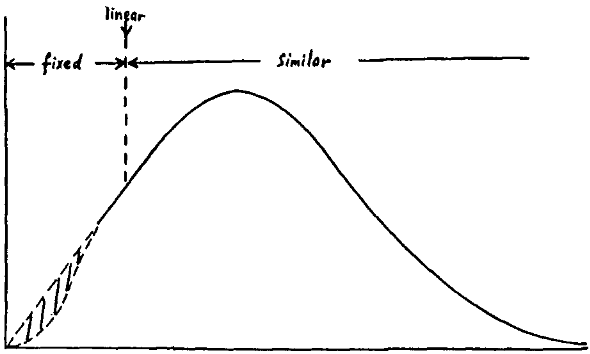


FIG. 1. Spectrum during early period.

where $R_{\lambda 0}$ is the initial Reynolds number

$$R_{\lambda 0} = \lim_{t\to 0} (u'\lambda/\nu), \tag{20}$$

and D_0 is a quantity proportional to D, defined by

$$D_0 = \lim_{t\to 0} (u'^2\lambda^2/\nu). \tag{21}$$

A formula of the type of Eq. (21) for the diffusion coefficient has been suggested previously (1937) by the senior author.[7] The law of decay (19) was

[7]Th. v. Kármán, J. Ae. Sci. 4, 131 (1937).

given by the junior author.[5] The role of the relatively invariant low frequency components has also been discussed by Heisenberg and Batchelor.

By using the law (19), one can see from (14) that the linear range of the spectrum would exist essentially for small values of t/T^*. This sets a limit to the period of validity of the law of decay (19). One can easily see the same limitation from the law (19) itself.

It can now be seen that the condition (12) is essentially the requirement that the Reynolds number is large (see (21)). It should be noted however that the existence of the $\kappa^{-5/3}$-range is not essential in the above discussion of the decay process. Hence, the Reynolds number need not be large, in order that the law (19) holds. In fact, for small initial Reynolds numbers, the Reynolds number at the end of the early period may be so small that the final period already sets in. This explains the agreement obtained by the junior author for almost the whole process of decay in comparing his theory with the experiments of Batchelor and Townsend.[8]

(II) *The intermediate stage.* For large Reynolds numbers, after the disappearance of the linear part of the spectrum, the scales L and L^* become of the same order of magnitude, and it may be expected that the bulk of turbulent energy of scale L shares the behavior of the large eddies of scale L^*. The ratios L/L^* and V/V^* are expected to be constants. These conditions lead at once to the law of decay discussed by the senior author.[4] It is characterized by $d\lambda^2/d(\nu t) = 7$.

During this period, the spectrum at low and medium frequencies depends only on the parameters J_0 and ν, and must therefore be of the form

$$F = J_0 \kappa^4 \Phi(\kappa L), \tag{22}$$

where $\Phi(\kappa L)$ behaves as $(\kappa L)^{-17/3}$ for $\kappa L \gg ?$, and approaches unity when $\kappa L \to 0$. An interpolation formula for Φ has been suggested and checked by correlation measurements by the senior author.

The diffusion coefficient in this range is easily seen to be proportional to $u'^2\lambda^2/\nu$. Hence, the condition (12) is again that the Reynolds number should be large. When the Reynolds number becomes very small, the scales η and L are of the same order, so that there is only one scale for all frequencies. We then approach a complete similarity, and are at the beginning of the final period. With reference to (12), we see that it should happen when R_λ is of the order of unity. According to the experiments of Batchelor and Townsend, the final period sets in at $R_\lambda \sim 5$.

The intermediate stage is very long, if the initial Reynolds number is very large. It begins with some value of R_λ close to $R_{\lambda 0}$. During this period, R_λ changes according to the power law $t^{-3/14}$. Although the supposed origin of

---

[8] G. K. Batchelor and A. A. Townsend, Proc. Roy. Soc. London, A **194**, 527 (1948).

time in this formula is unknown, it must be before the beginning of the early period, since the slope of the λ^2 *versus* νt curve decreases. Thus, R_λ become of the order of unity only when t is of the order $T^*R_{\lambda 0}^{14/3}$. One expects therefore to find an intermediate stage many times the early period for high initial Reynolds numbers.

The above predictions are based on some simple hypotheses and physical picture, and should be confirmed experimentally. Unfortunately, there does not seem to be any experimental data available for sufficiently high Reynolds numbers and over a sufficiently long period. Most of the decay measurements at high Reynolds numbers hardly extend beyond the early period, when the law of decay (19) is quite adequate. Also, it must be kept in mind that the above discussions hold only for an infinite field of turbulence. In an actual experiment, the scale of the apparatus might become comparable with the scale of turbulence. In such cases, the significance of Loitsiansky's invariant becomes uncertain.

INDEX